A solution to the Lottery Paradox through Fractional Semantics

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Abstract

The Lockean Thesis and the Lottery Paradox have generated significant discussions within the context of classical logic. In this paper, we aim to provide a solution to this relationship by utilizing Fractional Semantics. The approach offered by Fractional Semantics is flexible enough to address the Lottery Paradox without using probability or non-classical logic. Instead, Fractional Semantics introduces a distinct type of semantics for Classical Logic, which proves to be an effective tool for resolving the paradox.

Using Fractional Semantics to solve the Lottery Paradox offers several benefits. First and foremost, it maintains the conjunction between beliefs while still utilizing Classical Logic. Furthermore, it allows for a more nuanced approach to the problem, offering greater insight into the nature of truth and falsehood within a belief system.

Our research demonstrates that Fractional Semantics provides a viable solution to the Lottery Paradox and highlights the value of considering different approaches to logical reasoning. By incorporating a more flexible and nuanced framework for understanding the nature of truth and belief, we can develop more effective strategies for addressing complex problems in the real world.[1]

Introduction

The purpose of this paper is to provide a solution to the Lottery Paradox, which is a well-known issue in classical logic. The paradox is closely linked to the Lockean Thesis, which proposes that an agent can choose a rational

[1]**Keywords**: classical propositional logic, proof theory and constructive mathematics, many-valued logics, logics of knowledge and belief, Lottery Paradox, Lockean Thesis, beliefs and probability.

number within the interval [0,1] to serve as a truth threshold for the semantic interpretation of a formula. This threshold determines when an agent considers a proposition to be true. However, a paradox arises when this threshold is applied to a lottery, resulting in the Lottery Paradox.

When using the Lockean Thesis to determine our beliefs in a lottery, we would believe that each ticket is not the winning one. However, when we construct a conjunction of all our beliefs, stating that no particular ticket will win, this leads to a contradiction since one winning ticket must exist. One proposed solution to this problem is to reject the closure of beliefs under conjunction [11], but this is a strong thesis.

Our main idea is that Fractional Semantics can provide a solution to the Lottery Paradox without requiring the rejection of the closure of beliefs under conjunction. By utilizing Fractional Semantics, we can represent an agent's beliefs as a set of truth values between 0 and 1 rather than a single threshold value. This allows for a more nuanced and flexible representation of an agent's beliefs, which can help to untangle this paradoxical situation.

In the first section, we will provide a brief overview of the basic concepts of fractional semantics [13, 15]. In the second section, we will examine how Fractional Semantics can be expanded through the incorporation of beliefs, which were initially introduced in [2]. Finally, in the last section, we will discuss a method that may be useful in resolving the Lottery Paradox, thanks to the developed theoretical platform.

1 Fractional Semantics' framework

Fractional semantics [13, 15] for classical logic is a type of multi-valued semantics that operates based on pure proof theoretic considerations, where truth-values are rational numbers in the interval [0,1]. Unlike classical Boolean interpretation, fractional semantics breaks the symmetry between tautologies and contradictions. It assigns values that measure the *level* of contradiction of a formula. Thus, it can be used as a measure of whether a given proposition is closer to a tautology or a contradiction.

To allow for a fractional interpretation of its formulas, fractional semantics requires an appropriate proof-theoretic platform, namely a decidable logic $\mathcal{L}$ that can be displayed in a sequent system **S**. Fractional semantics is obtained by focusing on the axiomatic structure of proofs expressed in Kleene's one-side sequent system $GS4$ [10, 17]. The system has the following rules:

$$\frac{}{\vdash \Gamma, p, \overline{p}} \ (ax.)$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \ (\vee) \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B} \ (\wedge)$$

Negation is inductively defined by different atomic formulas p and $\overline{p}$, where $\overline{p}$ indicates the negation of p. The interpretation of a formula is the result of the ratio between the number of identity top-sequents $(\Delta, p, \overline{p})$ out of the total number of top-sequents occurring in any of its proofs. Weakening and contraction are dropped while cut rule has the form:

$$\frac{\vdash \Gamma, p \quad \vdash \overline{p}, \Delta}{\vdash \Gamma, \Delta} \ (cut)$$

In order to give a fractional interpretation a counterpart is needed, namely $\overline{\overline{GS4}}$, that is the $GS4$ calculus maximally extended:

Definition 1.1 ($\overline{\overline{GS4}}$)**.** *$\overline{\overline{GS4}}$ is defined as $GS4$ calculus maximally extended, adding the complementary axiom schema which enables the introduction of whatsoever consistent clause $\vdash \Delta$. The system $\overline{\overline{GS4}}$ is deductively trivial, i.e. anything can be derived in the system.*

Definition 1.2 ($[\![\Gamma]\!]$)**.** *$[\![\Gamma]\!]$, for each multiset Γ, indicates the fractional semantics value of Γ.*

In $\overline{\overline{GS4}}$, for instance, $\vdash p \rightarrow (q \wedge p)$ translated in $GS4$ as $\vdash \overline{p} \vee (q \wedge p)$ is derivable:

$$\dfrac{\dfrac{\dfrac{}{\vdash \overline{p}, q} \ \overline{ax.} \qquad \dfrac{}{\vdash \overline{p}, p} \ ax.}{\vdash \overline{p}, q \wedge p} \ (\wedge)}{\vdash \overline{p} \vee (q \wedge p)} \ (\vee)$$

In classical logic the value of this sequent would be 0, but in fractional semantics it is possible to assign a value based on the number of tautological clauses out of two axioms in total, i.e., $[\![\overline{p} \vee (q \wedge p)]\!] = 1/2 = 0.5$, because of the presence of $\vdash \overline{p}, p$. Stability guarantees that any other decomposition of this sequent will always return the value 0.5. It is possible to give a formal definition of top-sequents axioms:

Definition 1.3 (Top-sequents axioms)**.**

$top^1(\pi)$: denotes the multiset of all and only π's top-sequents introduced by an identity axiom;

$top^0(\pi)$: denotes the multiset of all and only π's top-sequents introduced by a complementary axiom, in other words, those axioms that are not tautological.

Any formula A can be interpreted as the ratio between the number of identity top-sequents (sequents introduced by the standard axiom) out of the total number of top-sequents.

$$[\![A]\!] = \frac{top^1(\pi)}{top^1(\pi) + top^0(\pi)}$$

Example 1.1. *Let's take another example to make the process clearer: let's take $\vdash (p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})$ and decompose it:*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash p, q}(\overline{ax.})}{\vdash p \vee q}(\vee) \qquad \dfrac{\dfrac{}{\vdash p, \overline{p}}(ax.)}{\vdash p \vee \overline{p}}(\vee)}{\vdash (p \vee q) \wedge (p \vee \overline{p})}(\wedge) \qquad \dfrac{\dfrac{}{\vdash \overline{r}}(\overline{ax.}) \qquad \dfrac{}{\vdash \overline{t}}(\overline{ax.})}{\vdash (\overline{r} \wedge \overline{t})}(\wedge)}{\vdash (p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})}(\wedge)$$

This proof contains one identity axiom out of four axioms in total and this means that:

$$[\![(p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})]\!] = \frac{1}{4} = 0.25$$

The final value of the sequent A will be 1/4 and it is the same for every decomposition of the considered sequent (Figure 1). Fractional semantics is able to keep trace of the number of contradictions out of the total number of axioms.

Figure 1: The value of $[\![A]\!]$.

It is possible to define the fractional semantics in terms of a multi-valued logics, made by this definition:

Definition 1.4 (Top-sequents)**.** *Top-sequents represent the number of the leaves of the proof as defined in Definition 1.3 and $[\![\vee\Gamma]\!]$ represents the value assigned to the multiset Γ where only $\vee-$applications appear.*

$top^1(\pi)$ *: let's call this m*

$top^0(\pi)$ *: let's call this n*

$[\![\vee\Gamma]\!]$ *is* $\frac{m}{n} \in [0,1]$

From this definition it is possible to give general rules:

$$\dfrac{}{\vdash^{1}_{1} \Gamma, p, \overline{p}}\ (ax.) \qquad\qquad \dfrac{}{\vdash^{0}_{1} \Delta}\ (\overline{ax.})$$

$$\dfrac{\vdash^{m}_{n} \Gamma, A, B}{\vdash^{m}_{n} \Gamma, A \vee B}\ (\vee) \qquad\qquad \dfrac{\vdash^{m_1}_{n_1} \Gamma, A \qquad \vdash^{m_2}_{n_2} \Gamma, B}{\vdash^{m_1+m_2}_{n_1+n_2} \Gamma, A \wedge B}\ (\wedge)$$

This method will permit to keep track of the value at every stage of the proof.

Example 1.2. *Let's take the same sequent considered before in example 1.1, but now using this decorated derivation.*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} p, q}\ (\overline{ax.})}{\vdash^{0}_{1} p \vee q}\ (\vee) \qquad \dfrac{\dfrac{}{\vdash^{1}_{1} p, \overline{p}}\ (ax.)}{\vdash^{1}_{1} p \vee \overline{p}}\ (\vee)}{\vdash^{1}_{2} (p \vee q) \wedge (p \vee \overline{p})}\ (\wedge) \qquad \dfrac{\dfrac{}{\vdash^{0}_{1} \overline{r}}\ (\overline{ax.}) \qquad \dfrac{}{\vdash^{0}_{1} \overline{t}}\ (\overline{ax.})}{\vdash^{0}_{2} (\overline{r} \wedge \overline{t})}\ (\wedge)}{\vdash^{1}_{4} (p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})}\ (\wedge)$$

The main difference between Example 1.1 and 1.2 is that in 1.2 is possible to read the fractional semantics value each time a rule is introduced. For example the fractional semantics value of $\vdash (p \vee q) \wedge (p \vee \overline{p})$ will be $1/2 = 0.5$ as it's easily noticeable in the proof.

2 Framing beliefs into fractional semantics

Now that the general framework is presented, it is possible to see how to consider beliefs in the fractional semantics for classical logic, firstly appeared in [2]. The idea is simple: a set of axioms, let's say B, that are considered true by an agent, is added to the system and every proposition in it is considered as true as tautologies. The philosophical idea behind this process is that an agent usually considers true their own beliefs.

Beliefs would be considered as deductively closed: this means that every deduction made using true beliefs will be considered true and this also means that the agent is a deductively ideal one. It is interesting to see what happens if the fractional semantics is provided with a system that is able to manage these new axioms, because it will be possible to obtain values greater than the ones that fractional semantics usually permits.

The idea for this kind of expansion was born thanks to the first of the three methods that Makinson used in [12] to bridge the gap between classical and non-monotonic logic, made by adding background assumptions. This kind of method was called *pivotal-assumption consequence* and permitted to infer more than classical logic permits thanks to a set of axioms that are added to the premises in every deduction.

Fractional semantics for classical logic updated with a set of new beliefs is different from pivotal-assumption consequence because of two main reasons. The first one is that Makinson used a classical two-valued semantics, whereas fractional semantics is a multi-valued interpretation. On the one hand *pivotal assumption consequence* would assign the value 0 if at least one of the axioms is neither a proper axiom nor a belief, on the other hand fractional semantics is provided by a system able to assign values greater than 0 when one of the top sequents is a tautology or a belief. The second reason is that, although both of them were born thanks to syntactical techniques, Makinson used an Hilbert-style approach, while fractional semantics uses the Gentzen-style one.

In order to add beliefs to the system, they must be atomic, if they are not, they must be decomposed.

Definition 2.1 ($GS4_B$)**.** *Let* $GS4$ *as defined earlier,* $GS4_B$ *is defined as* $GS4$ *with a set of new axioms, namely* $B = b_1, \ldots, b_n$*, that represents a non-contradictory set of beliefs of an agent. Each* b_i *(*$1 \leq i \leq n$*) must be atomic.*

Definition 2.2 ($\vdash_B$)**.** *If* $\vdash$ *is the closure relation of classical logic,* $\vdash_B$ *is defined as the closure relation of* $GS4_B$*.*

Remark 2.1. *Makinson [12] pointed out the problems arising when new axioms are added to the system. In fact, substitution is no longer acceptable when the system has the possibility to manage new non-logical axioms. The same result, even if it is not cited by Makinson, is due to the fact that classical logic is a Post-complete system and this means that, once a nontautological formula is added to the system, the new system will be inconsistent, unless structurality is dropped. The system loses the structurality because structurality and consistency are mutually excluding properties in classical logic with extra-logical axioms [14].*

Now it's possible to go further into the formalization of the system: let's define the top sequent made from beliefs added to the system.

Definition 2.3. *$top^b(\pi)$: denotes the multiset of all and only π's top-sequents introduced by a belief.*

$$[\![A]\!]_B = \frac{top^b(\pi) + top^1(\pi)}{top^b(\pi) + top^1(\pi) + top^0(\pi)}$$

Example 2.1. *Let's see an example taking the same sequent seen in example 1.1, but adding now the belief $\vdash_B (p \vee q) \wedge \overline{u}$. The first thing to do, in order to add the belief, is to decompose it and add that to the belief set.*

$$\dfrac{\dfrac{\dfrac{}{\vdash_B p, q}\,(b_1)}{\vdash_B p \vee q}\,(\vee) \qquad \dfrac{}{\vdash_B \overline{u}}\,(b_2)}{\vdash_B (p \vee q) \wedge \overline{u}}\,(\wedge)$$

From this it is possible to add the two beliefs $b_1 = p, q$ and $b_2 = \overline{u}$ to the belief set.

Let's take the same decomposition seen in example 1.1: it is possible to see where a belief is added to the system, namely b_1, because $\overline{u}$ doesn't appear in the sequent.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_B p, q}\,(b_1)}{\vdash_B p \vee q}\,(\vee) \qquad \dfrac{\dfrac{}{\vdash_B p, \overline{p}}\,(ax.)}{\vdash_B p \vee \overline{p}}\,(\vee)}{\vdash_B (p \vee q) \wedge (p \vee \overline{p})}\,(\wedge) \qquad \dfrac{\dfrac{}{\vdash_B \overline{r}}\,(\overline{ax.}) \qquad \dfrac{}{\vdash_B \overline{t}}\,(\overline{ax.})}{\vdash_B \overline{r} \wedge \overline{t}}\,(\wedge)}{\vdash_B (p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})}\,(\wedge)$$

This proof contains one identity axiom, one belief and two complementary axioms, so $[\![A]\!]_B$ is:

$$[\![A]\!]_B = \frac{top^b(\pi) + top^1(\pi)}{top^b(\pi) + top^1(\pi) + top^0(\pi)} = \frac{1+1}{1+1+2} = \frac{2}{4} = 0.5$$

Like in example 1.2, it is possible to see the same tree with multi valued system, adding a new rule:

$$\dfrac{}{\vdash^{1}_{1\,B} B}\,(\overline{b_i})$$

Where B denote the set of belief, $B = b_1, \ldots, b_n$.

Example 2.2. *Now it's possible to see Example 2.1 with the multi valued system.*

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\vdash^{1}_{1\,B} p, q}\,(b_1)}{\vdash^{1}_{1\,B} p \vee q}\,(\vee)
\qquad
\dfrac{\dfrac{}{\vdash^{1}_{1\,B} p, \overline{p}}\,(ax.)}{\vdash^{1}_{1\,B} p \vee \overline{p}}\,(\vee)
}{\vdash^{2}_{2\,B} (p \vee q) \wedge (p \vee \overline{p})}\,(\wedge)
\qquad
\dfrac{
\dfrac{}{\vdash^{0}_{1\,B} \overline{r}}\,(\overline{ax.})
\qquad
\dfrac{}{\vdash^{0}_{1\,B} \overline{t}}\,(\overline{ax.})
}{\vdash^{0}_{2\,B} (\overline{r} \wedge \overline{t})}\,(\wedge)
}{\vdash^{2}_{4\,B} (p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})}\,(\wedge)
$$

The only difference between Example 1.2 and 2.2 can be seen in the substitution with the value 1 instead of the 0 for the top axiom $\vdash p, q$. In Figure 2 it is possible to see how the value of A changed when was considered in the fractional semantics framework without beliefs and when one belief is added.

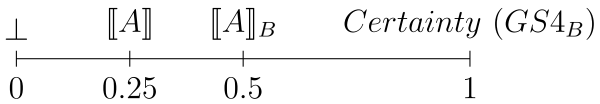

Figure 2: The values of $[\![A]\!]$ and $[\![A]\!]_B$.

It is worth noting that if this sequent was considered in classical logic, it would have different values changing by the value of the atomic formulas, but it would assume value 0 or 1. Something similar happens in Makinson pivotal assumption consequence, also if the belief set is the same that we have defined earlier, because a two valued logic is there considered. For completeness of exposition we state these theorems that are proved in [2] and [14].

Theorem 2.1. *For any context* Γ *and a formula* A*, such that* A *is not contradictory with the set* B*,* $[\![\bigvee \Gamma \vee A]\!]_B \geq [\![\Gamma]\!]_B$.

Theorem 2.2 (Strong cut elimination of $GS4_B$)**.** *The cut rule is redundant when added to* $GS4_B$.

Theorem 2.3 (Uniqueness of axiomatization in $GS4_B$)**.** *For any cluster of axioms in the set of beliefs* B *the axiomatization is unique.*

3 Fractional semantics, Lockean Thesis and the paradox of Lottery

An interesting application of fractional semantics for classical logic, as stated in the introduction, can be found in the solution of a traditional problem: the Lottery Paradox. The Lottery Paradox is closely related to the Lockean Thesis, which defines how classical logic aligns with beliefs.

In brief, the Lockean Thesis asserts that an agent can select a number within the interval [0,1] for the semantic interpretation of a formula that works as a threshold for truthfulness. This threshold imposes a level above which an agent considers a certain proposition or set of propositions as true. For example, if an agent assigns a threshold value of 0.9, any sentence with a value greater than 0.9 will be considered as true as a tautology, while a value less than 0.9 will be regarded as false. The term *Lockean thesis* does not refer to Locke himself proposing it, but to his discussion of probability in An Essay Concerning Human Understanding:

> But another Man who never took the pains to observe the Demonstration, hearing a Mathematician, a Man of credit, affirm the three Angles of a Triangle, to be equal to two right ones, assents to it: i.e. receives it for true.[2]

Beliefs are related to the level of assent that one agent can give to another and so to those beliefs that are not 100% true, but in which the agent still has a high degree of confidence in them.

> Being that which [the probability] makes us presume things to be true, before we know them to be so.[3]

Here probability is treated as the level of assent in a certain proposition, in fact the renewed version of the Lockean Thesis is formulated as it follows in [3]:

> It is epistemically rational for us to believe a proposition just in case it is epistemically rational for us to have a sufficiently high degree of confidence in it, sufficiently high to make our attitude towards it one of belief.

[2] John Locke, *An essay concerning human understanding* (1690) 4^{th} book, chapter XV, §1

[3] ivi. 4^{th} book, chapter XV, §3

The Lockean Thesis has many important positive aspects. For instance, it implies that even a logically ideal agent whose degrees of confidence satisfy the axioms of probability theory can rationally believe each of a large body of propositions that are jointly inconsistent. This can be beneficial in situations where the agent has incomplete information or where there are various sources of uncertainty.

However, as we will see, beliefs under the Lockean Thesis are not closed under conjunction because inconsistent beliefs with varying degrees of confidence are possible. This means that an agent can simultaneously hold contradictory beliefs with different levels of confidence. One criticism of the Lockean Thesis is that it does not provide clear guidance on how to select the threshold. Determining which propositions an agent considers to be true or false is crucial, and the threshold can also affect the coherence and consistency of an agent's beliefs. However, the Lockean Thesis does not specify how an agent should select the threshold, and some argue that it can seem arbitrary.

Despite its strengths, the Lockean Thesis faces a significant challenge in the form of the Lottery Paradox. The paradox can appear straightforward at first glance, but it comes into conflict with the Lockean Thesis.

The Lottery Paradox Let's consider a fair 1000-ticket lottery that has only one winning ticket. A perfectly rational agent knows that each ticket has a probability of 999/1000 of not winning. Thus, it is rational for the agent to accept that each ticket will not win because this probability is greater than her Lockean threshold. This reasoning can be extended to every other ticket in the lottery, leading to the conclusion that somehow every ticket will not be the winning ticket. However, the lottery is fair, so the conjunction of all these statements has to be false, rather than true as it appears.

This seemingly simple paradox highlights an interesting problem that can be formalized in modal logic through the Barcan formula (BF). The BF is problematic when considered with the lottery paradox, whereas the converse Barcan formula (CBF) is not problematic.

$$\forall x \Box F(x) \rightarrow \Box \forall x F(x) \tag{BF}$$

$$\Box \forall x F(x) \rightarrow \forall x \Box F(x) \tag{CBF}$$

The converse Barcan formula is not problematic in this case, as it simply states that if it is necessary for all x to have the property F, then each individual x must have the property F. Therefore, if it is necessary for every ticket to be a losing one, then each individual ticket must be a losing one.

However, the Barcan formula itself presents a problem in the case of the lottery paradox. It states that if every x is necessarily F, then it is necessary that every x is F. In the context of the lottery paradox, if it is necessary that each individual ticket is a losing one, then it is necessary that every ticket is a losing one. This leads to a contradiction: each individual ticket considered alone is a losing ticket, but the conjunction of all losing tickets couldn't be true, as there must be at least one winning ticket.

One proposed solution to this problem, as discussed in sources such as [8] and [11], is to reject the closure of beliefs under conjunction. This is a strong thesis, as it implies that inconsistent beliefs can be rational. The authors propose that this is due to the fact that beliefs are not completely certain and can change over time, allowing for the possibility of holding multiple inconsistent beliefs simultaneously.

However, it may be desirable to maintain belief closure under conjunction. Through Fractional Semantics, it is possible to achieve this while using classical logic. To apply this approach to the lottery paradox, we can represent each ticket as a proposition, denoting whether or not it is the winning ticket. Let p_n represent the proposition that ticket n will win, where $1 \leq n \leq 1000$, and let p_i represent the winning ticket.

To simplify the problem, but without loss of generality, we can assume that the first and last tickets are not winning tickets and enumerate the tickets as follows:

$$p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_{1000}$$

We can then consider the negations of these propositions, $\overline{p_1}, \ldots, \overline{p_i}, \ldots, \overline{p_{1000}}$, and represent them in a tree.

By using the Fractional Semantics expansion presented in section 2, we can assign a truth value chosen between 0 and 1 to each proposition, indicating whether the proposition is a belief or not. In this case, the value of each non-winning ticket will be 1, and the value of the winning ticket will be 0, because $\overline{p_i}$ is false, i.e., that p_i will win. This system preserves classical logic and allows us to maintain belief closure under conjunction while also resolving the paradox.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash^{1}_{1\,B}\overline{p_1} \qquad \cdots \qquad \vdash^{1}_{1\,B}\overline{p_{i-1}}}{\vdash^{i-1}_{i-1\,B}\overline{p_1}\wedge\cdots\wedge\overline{p_{i-1}}} \qquad \dfrac{}{\vdash^{0}_{1\,B}\overline{p_i}}}{\vdash^{i-1}_{i\,B}\overline{p_1}\wedge\cdots\wedge\overline{p_{i-1}}\wedge\overline{p_i}} \qquad \dfrac{}{\vdash^{1}_{1\,B}\overline{p_{i+1}}}}{\vdash^{i}_{i+1\,B}\overline{p_1}\wedge\cdots\wedge\overline{p_{i-1}}\wedge\overline{p_i}\wedge\overline{p_{i+1}}} \qquad \cdots \qquad \dfrac{}{\vdash^{1}_{1\,B}\overline{p_{1000}}}}{\vdots}}{\vdash^{999}_{1000\,B}\overline{p_1}\wedge\cdots\wedge\overline{p_{i-1}}\wedge\overline{p_i}\wedge\overline{p_{i+1}}\wedge\cdots\wedge\overline{p_{1000}}}
$$

Fractional Semantics deals perfectly with this paradox, providing a very simple solution to this problem. In fact it is easy to see that in Fractional Semantics it is not useful to have a threshold, because in every moment is possible to control the value of a proposition and also the history of how that value becomes itself thanks to the proof tree. The final value will be 999/1000 and means that 999 parts of the conjunction are true out of the 1000 joints and this coheres with the fact that one ticket must be the winning one.

Fractional semantics and probability Must be stressed here that the fractional semantics value is not a probabilistic one, in fact the probability measure of the conjunction is made by the conditionalization formula as pointed out in [7]. In this formalism $\mathcal{P}$ indicates the probability, $(\mathcal{P}(\overline{p_1})|\mathcal{P}(\overline{p_2}))$ indicates the probability that the event $\overline{p_2}$ happens once $\overline{p_1}$ happened. This is called conditionalization because it is the probability that a certain event happens if another happened, *conditioning* the final probability and it is calculated by the following formula.

$$\mathcal{P}(\overline{p_1}\wedge\overline{p_2}) = \mathcal{P}(\overline{p_1})\cdot(\mathcal{P}(\overline{p_1})|\mathcal{P}(\overline{p_2}))$$

This means that, once $\overline{p_1}$ is realized, also $\overline{p_2}$ realizes, so:

$$\mathcal{P}(\overline{p_1}\wedge\overline{p_2}) = \frac{999}{1000}\cdot\frac{998}{999} = \frac{998}{1000}$$

The meaning of the fractional semantics value is that the conjunction is true for 999 of the joints and false for only one of them. Where the probability or classical logic assigns value 0, fractional semantics helps us to understand that, also if the final conjunction results false in classical logic, actually all but one of the propositions are true and this result is not explicit in classical framework.

4 Conclusions

The aim of this research is to establish a connection between logic and the real world while maintaining consistency and decidability. Specifically, we focus on addressing the Lottery Paradox, a problem that has troubled philosophers and logicians.

The traditional logic approach fails to capture the nuances of truthfulness and falsity that are present in the real world. In contrast, Fractional Semantics is capable of making distinctions between various levels of contradiction, allowing for a more nuanced approach to proof theory while retaining rigor.

We introduce the system $GS4_B$, which combines beliefs and logical axioms, highlighting the importance of considering both together. Fractional Semantics offers a helpful framework to reason about uncertainty, which is inherent in beliefs. Our research shows that Fractional Semantics provides a solution to the Lottery Paradox, a seemingly simple problem with a complex solution.

It's important to note that fractional semantics behaves differently from classical interpretation and probability. By highlighting the differences between these approaches, we can explore how they might work together to provide a more refined understanding of logical derivations that involve beliefs. Overall, this research offers a significant contribution to the field of logic by demonstrating the potential of Fractional Semantics in bridging the gap between theory and practice.

Further researches The current implementation of fractional semantics has already shown promising results, but there is potential for further expansion. One such area of exploration is the consideration of restrictions on the set of beliefs as Makinson does in [12]. Fractional Semantics could be applied to non-monotonic logics or non-classical logics, offering a more nuanced approach to reasoning with uncertain or incomplete information.

In our investigation of the Lottery Paradox, we observed that the use of fractional semantics led to richer insights than probability theory alone. By analyzing the values of true and false conjunct propositions, we gained a deeper understanding of the paradox. However, we must also acknowledge that there is still work to be done to integrate probability, Fractional Semantics, and Belief Revision.

In particular, there is a need to develop methods that allow these approaches to work together seamlessly. By doing so, we can refine our understanding of how beliefs and uncertainty impact logical derivations. Ultimately, this would provide a more comprehensive and nuanced framework for reasoning about complex problems in the real world.

Overall, the potential of Fractional Semantics to enhance our understanding of uncertain or incomplete information is significant. With further exploration and development, it has the potential to transform how we approach logical reasoning and decision making in a wide range of applications.

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