Foundations of LLCM: Labelled Lambek Calculus for Music Analysis

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Abstract. This paper presents an application of Lambek Calculus, a sequent calculus for categorial grammar, to music analysis. To this end, we propose a *Labelled Lambek Calculus for Music* (LLCM), where a label represents tonality information. In LLCM, each adjacent category represents a chord interpretation. When combined, they form a cadential category. We have enhanced the system with a rigorous and internally consistent framework that clarifies long-distance dependencies and provides a more explicit representation of relationships across different tonalities, including tonal shifts. A key innovation in LLCM is the introduction of a method for calculating the "depth" of a harmonic analysis. This measure corresponds to the complexity of chord progressions, enabling analysts to objectively compare different harmonic sequences based on their structural intricacy.

Keywords: Lambek Calculus · Music Analysis · Chord Analysis · Proof Theory.

1 Introduction

In the categorial grammar of natural language, we can compose a grammatical sentence combining parts of speech, *categories* in general, by the following (/) and (\):

$$Y/X : \text{biting } X \text{ from the right-hand side, } Y$$
$$X\backslash Y : \text{biting } X \text{ from the left-hnad side, } Y.$$

For example, a verb phrase (VP) bites a noun phrase (NP) from the left-hand side as a subject to be a sentence (S) so that $VP = NP\backslash S$, while a determiner (Det) bites a noun from the right-hand side to be an NP so that $Det = NP/N$. Therefore, for a sequence of

$$\text{A} : NP/N, \ \text{man} : N, \ \text{walks} : NP\backslash S,$$

we can compose a sentence as

$$\dfrac{\dfrac{\text{A: } NP/N \qquad \text{man: } N}{\text{A man: } NP} \qquad \text{walks: } NP\backslash S}{\text{A man walks: } S}$$

In [7], the categorial grammar was applied to analyze jazz chord sequences.[3] We would like to extend this formalism to the more rigorous Lambek calculus, that is, a Gentzen style sequent calculus [6, 8] for the categorial grammar, improving the preceding [1–3] with [19]. In this paper, given a sequence of categories Γ, we discuss whether we can obtain category S, represented as $\Gamma \vdash S$, limiting the right-hand side of '$\vdash$' intuitionistically to a single term at every step. In order to do this, we are required to extend the existing Lambek Calculus [14, 15] in the following respects.

- We employ labelled sequent calculus for the categorial grammar, and name it Labelled Lambek Calculus, where a label represents a tonality.
- In this paper, we discuss the progression from left to right of the chords, so we only consider ($\backslash$), omitting ($/$).
- The syntactic *norm* for music is much weaker than that of natural language; a sentence must compose one proof tree, whereas a music piece could be a sequence of trees. Therefore, we introduce the tree concatenation ($\succ$).

In the following sections, we present the foundational definitions and sequent rules for the Labelled Lambek Calculus for Music Analysis (LLCM).

2 Preliminaries

As input for our system the classic Berklee Chord Names (Table 1) is used.

XMA^7	Major seventh.
Xm7	Minor seventh.
XmMA^7	Minor chord with major seventh.
Xm$7^{\flat 5}$	Semi diminished or minor.
Xdim7	Diminished chord.
Xsus4	Suspended fourth.
X7	Dominant seventh.
$\text{X}^{\sharp 5}$	Augmented fifth.

Table 1. Berklee Chord Names

F	$\Rightarrow \text{IV}^C, \text{I}^F, \text{V}^{B\flat}, \cdots$
G	$\Rightarrow \text{I}^G, \text{IV}^D, \text{VII}^a, \cdots$
B♭	$\Rightarrow \text{IV}^F, \cdots$
C7	$\Rightarrow \text{V7}^F, \cdots$
$\vdots$	$\vdots$

Table 2. Lexical interpretation

The key, i.e. the tonality, will be indicated in general with a low case greek letter, such as $\alpha, \beta, \ldots$ and in the analysis with a low case letter for minor tonalities and upper case for major tonalities, for example 'C' stands for C major, 'a' stands for A minor, and so on.

[3] Categorial grammar has the same generative power as context-free grammar (CFG), and the tree structure of CFG has been applied in music [17, 18] also.

The degree will generally be indicated with $x, y, \ldots$, and in analysis, it will be represented by a Roman numeral associated with the function, such as IIIm7, $\mathrm{II}\textsc{ma}^7, \ldots$, unless the function is clear from the tonal context. In such cases, we will simply use the Roman numeral for simplicity.

In labelled sequent calculus, it is common to denote propositions with the symbol : connecting the label α and the atom x, as in $\alpha\colon x$. However, to save space and improve readability of the derivation trees, we opted for the notation x^α, which preserves the same logical meaning. The lexicon provides a list where a Berklee chord is looked up and it can be *interpreted* in multiple ways, as in Table 2.

3 Definitions of the LLCM-rules

A sequent is such a form as

$$\mathbb{A} \vdash z^\gamma,$$

where the left-hand side of $\vdash$ is the antecedent of the sequent and the right-hand side is the consequence. A bold Latin letter ($\mathbb{A}$, $\mathbb{B}$, $\ldots$) is employed to denote a sequence of labelled terms. For example,

$$\mathbb{A} \stackrel{\text{def}}{\equiv} x^\alpha, y^\beta, \ldots.$$

An antecedent consists of labelled terms articulated by commas (,) which are interpreted as logical 'and' ($\wedge$) as usual. According to the requirement of categorial grammar, we do not allow such structural rules as *exchange*, *weakening* or *contraction*, and the right-hand side of $\vdash$ is restricted to be one term, to be intuitionistic. Hereafter, when a sequence of labelled terms shares a common label, we adopt the following convention.

$$\{x, y, \cdots\}^\alpha \stackrel{\text{def}}{\equiv} x^\alpha, y^\alpha, \ldots.$$

Definition 1 (Initial Sequent). *The initial sequent of an analysis will be the interpretation of a chord written similarly to a tautology in logic:*

$$\frac{\textit{Chord}}{x^\alpha \vdash x^\alpha}$$

It is possible having multiple interpretations of the same chord, based on the given lexicon, for example:

$$\frac{\mathrm{C}\textsc{ma}^7}{\mathrm{I}^C \vdash \mathrm{I}^C} \qquad \frac{\mathrm{C}\textsc{ma}^7}{\mathrm{IV}^G \vdash \mathrm{IV}^G} \qquad \frac{\mathrm{C}\textsc{ma}^7}{\mathrm{III}^a \vdash \mathrm{III}^a} \qquad \ldots$$

In music, events cannot simply be reversed in time without altering their meaning or function; thus, preserving the sequence of elements during analysis is essential. This temporal aspect highlights that the relationships between musical components are not interchangeable and must be considered in their specific order, as this affects both harmonic structure and expressive qualities.

Definition 2 ((\) Introduction). *The introduction rules are applied when a sequence of chords shares the same tonality, to form a chain that represents a harmonic progression. We employ the insertion of* (\) *only to the left-hand side of* ($\vdash$) *in LLCM*

$$\frac{\mathbb{A} \vdash x^{\alpha} \qquad \mathbb{B}, y^{\alpha}, \mathbb{C} \vdash z^{\beta}}{\mathbb{B}, \mathbb{A}, (x\backslash y)^{\alpha}, \mathbb{C} \vdash z^{\beta}} \; (\backslash_L)$$

Definition 3 (Accessibility Relation R). *The accessibility relation R^{α}_{β} shows a shift to a related tonality from α to β, changing accordingly the degree.*

$$\frac{\mathbb{A} \vdash x^{\alpha}}{\mathbb{A}, R^{\alpha}_{\beta} \vdash y^{\beta}} \; (R)$$

Since the framework is tonal, this type of modulation strictly follows the rules of tonal harmony. The value of y depends on the accessibility relation R from x as $y = f(R, x)$. When R is a shift to the dominant key, such as from C to G, we can express it as $f(R, x) = x + 4 \pmod 7$ or $f(R, x) = x - 3 \pmod 7$. Conversely, when R represents a shift to the subdominant key (the mirror of the dominant), we get $f(R, x) = x + 3 \pmod 7$ or $f(R, x) = x - 4 \pmod 7$, and so forth for the other functions. These operations apply to diatonic shifts and do not affect chromatic approaches, as R only governs diatonic functional relationships within the seven degrees of harmony.

To illustrate the utility of R, consider its usefulness when a chord assumes multiple functions. For example, in the cadence D7 G7 $\mathrm{C{\small MA}}^7$, the chord G7 serves a dual purpose: it acts as the tonic (I^G) in the key of G, while also serving as the dominant (V^C) in C major (cfr. Example 2). The accessibility relation R^G_C enables this functional shift, allowing G7 to bridge these two tonalities. The variety and usage of R depend on the genre and era of the target music, a topic we will explore further.

Definition 4 (The CONJ Rule). *When two sequents share the common consequence, we can employ the conjunction rule, which corresponds to such conjunctions as 'and' and 'or' in natural language.*

$$\frac{\mathbb{A} \vdash x^{\alpha} \qquad \mathbb{B} \vdash x^{\alpha}}{\mathbb{A}, \mathbb{B} \vdash x^{\alpha}} \; (CONJ)$$

Definition 5 (($\succ$)-rule). *To connect two adjacent phrases with unrelated tonalities, we define the ordered concatenation; '$(\mathbb{A} \vdash x^{\alpha}) \succ (\mathbb{B} \vdash y^{\beta})$'. For notational convenience, we write this in the style of a rule with one ($\vdash$), as follows. In accordance with ($\succ$), we articulate the antecedent by (;) (semicolon).*

$$\frac{\mathbb{A} \vdash x^{\alpha} \qquad \mathbb{B} \vdash y^{\beta}}{\mathbb{A}; \mathbb{B} \vdash x^{\alpha} \succ y^{\beta}} \; (\succ)$$

Note that '$x^\alpha \succ y^\beta$' is not a labelled term as x and y possess different labels, but in each sequent in '$\mathbb{A} \vdash x^\alpha$' and '$\mathbb{B} \vdash y^\alpha$' every term is uniquely labelled.

Actually, ($\succ$) represents our policy to *loosen* the syntactic norm of music; that is, a piece of music may not be a single proof tree but could be consecutive trees.

Definition 6. *By combining ($\succ$) and ($\backslash_L$), we introduce the (;) rules, which interact with the inner elements on either side of ($\succ$), provided that the tonalities match.*

$$\frac{\mathbb{A};\mathbb{B} \vdash x^\alpha \succ y^\beta \qquad z^\alpha \vdash z^\alpha}{\mathbb{A}, (x\backslash z)^\alpha; \mathbb{B} \vdash z^\alpha \succ y^\beta}\ (;_L) \qquad \frac{\mathbb{A};\mathbb{B} \vdash x^\alpha \succ y^\beta \qquad z^\beta \vdash z^\beta}{\mathbb{A}; \mathbb{B}, (y\backslash z)^\beta \vdash x^\alpha \succ z^\beta}\ (;_R)$$

In LLCM, the CUT rule is also admissible but we do not employ it in this paper.

Example 1 (The use of the ($\backslash_L$) *rule).* In the typical cadence: VI-II-V-I, the VI degree leads into the II chord, which further intensifies the progression towards the dominant. The V chord then resolves to the tonic I, completing the cadence. In LLCM is as following:

$$\cfrac{\cfrac{\cfrac{\cfrac{\text{Am7}}{\text{VI}^C \vdash \text{VI}^C} \qquad \cfrac{\text{Dm7}}{\text{II}^C \vdash \text{II}^C}}{\{\text{VI}, (\text{VI}\backslash\text{II})\}^C \vdash \text{II}^C}\ (\backslash_L) \qquad \cfrac{\text{G7}}{\text{V}^C \vdash \text{V}^C}}{\{\text{VI}, (\text{VI}\backslash\text{II}), (\text{II}\backslash\text{V})\}^C \vdash \text{V}^C}\ (\backslash_L) \qquad \cfrac{\text{C}\textsc{ma}^7}{\text{I} \vdash \text{I}}}{\{\text{VI}, (\text{VI}\backslash\text{II}), (\text{II}\backslash\text{V}), (\text{V}\backslash\text{I})\}^C \vdash \text{I}^C}\ (\backslash_L)$$

The idea of the accessibility relationship lies on the fact that sometimes a chord has a function if considered as the result of a cadence that appears before that, but maybe it has another function if it is considered as the first of a new chain.

Example 2 (The use of (R)). Let us consider a classical secondary dominant, i.e., D7, G7, C$\textsc{ma}^7$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\text{D7}}{\text{V}^G \vdash \text{V}^G} \qquad \cfrac{\text{G7}}{\text{I}^G \vdash \text{I}^G}}{\text{V}^G, (\text{V}\backslash\text{I})^G \vdash \text{I}^G}\ (\backslash_L)}{\{\text{V}, (\text{V}\backslash 1)\}^G, R^G_C \vdash \text{V}^C}\ (R) \qquad \cfrac{\text{C}\textsc{ma}^7}{\text{I}^C \vdash \text{I}^C}}{\{\text{V}, (\text{V}\backslash\text{I})\}^G, R^G_C, (\text{V}\backslash\text{I})^C \vdash \text{I}^C}\ (\backslash_L)$$

Reading the final sequent we can say that the cadence begins on the fifth degree of G major. Then we have a cadence on the tonic (first degree) of G major, which is related to the key of C major, changing its harmonic function. Afterward, there is a cadence from the fifth to the tonic (first degree) of C major, with the entire passage interpreted in the key of C major, which is the true tonality of this cadence.

Example 3 (The use of $(\succ)$). Let us consider two cadences belonging to different tonalities that do not have any relationship with each other. For example, the A-section of *All the Things You Are* by Jerome Kern contains a sequence of chords that belong to two different tonalities: A♭ and C. To make the tree structure more readable, we have omitted the first four chords (Fm7, B♭m7, and E♭7, A♭) which form a classic cadence on the tonic of A♭. While we have included the entire sequence in the derivation, we will focus on presenting only the essential part.

$$\dfrac{\dfrac{\dfrac{\vdots}{\{\mathrm{VI}, (\mathrm{VI}\backslash \mathrm{II}), (\mathrm{II}\backslash \mathrm{V}), (\mathrm{V}\backslash \mathrm{I})\}^{A\flat} \vdash \mathrm{I}^{A\flat}} \quad \dfrac{\mathrm{D}\flat\textsc{ma}^7}{\mathrm{IV}^{A\flat} \vdash \mathrm{IV}^{A\flat}}}{\{\mathrm{VI}, \cdots, (\mathrm{V}\backslash \mathrm{I}), (\mathrm{I}\backslash \mathrm{IV})\}^{A\flat} \vdash \mathrm{IV}^{A\flat}}\,{\scriptstyle (\backslash_L)} \quad \dfrac{\dfrac{G7}{\mathrm{V}^C \vdash \mathrm{V}^C} \quad \dfrac{\mathrm{C}\textsc{ma}^7}{\mathrm{I}^C \vdash \mathrm{I}^C}}{\{\mathrm{V}, (\mathrm{V}\backslash \mathrm{I})\}^C \vdash \mathrm{I}^C}\,{\scriptstyle (\backslash_L)}}{\{\mathrm{VI}, \cdots, (\mathrm{V}\backslash \mathrm{I}), (\mathrm{I}\backslash \mathrm{IV})\}^{A\flat}; \{\mathrm{V}, (\mathrm{V}\backslash I)\}^C \vdash \mathrm{IV}^{A\flat} \succ \mathrm{I}^C}\,{\scriptstyle (\succ)}$$

Reading this harmonic sequence we can say that we have a cadence in A♭ suddenly solved, as well as the one in C. They are connected by a relation R that doesn't change the harmonic function of the chords.

Example 4 (The use of (CONJ)). When a chord progression ends though it is followed by another progression that also concludes in the same tonality, two progressions are simply connected.

$$\dfrac{\dfrac{\dfrac{G7}{\mathrm{V}^C \vdash \mathrm{V}^C} \quad \dfrac{C\textsc{ma}^7}{\mathrm{I}^C \vdash \mathrm{I}^C}\,{\scriptstyle (\backslash_L)}}{\{\mathrm{V}, (\mathrm{V}\backslash \mathrm{I})\}^C \vdash \mathrm{I}^C}\,{\scriptstyle (\backslash_L)} \quad \dfrac{\dfrac{D\flat m7^{(\flat 5)}}{\mathrm{II}\flat^C \vdash \mathrm{II}\flat^C} \quad \dfrac{C\textsc{ma}^7}{\mathrm{I}^C \vdash \mathrm{I}^C}\,{\scriptstyle (\backslash_L)}}{\{\mathrm{II}\flat, (\mathrm{II}\flat\backslash \mathrm{I})\}^C \vdash \mathrm{I}^C}\,{\scriptstyle (\backslash_L)}}{\{\mathrm{V}, (\mathrm{V}\backslash \mathrm{I}), \mathrm{II}\flat, (\mathrm{II}\flat\backslash \mathrm{I})\}^C \vdash \mathrm{I}^C}\,{\scriptstyle (\mathrm{CONJ})}$$

Example 5 (Another use of (CONJ) called (CONJ$_2$)). The repetition can be considered as a kind of CONJ, e.g., the A section of *I Got Rhythm* or any typical Anatole progression: Am7, Dm7, G7, $\mathrm{C}\textsc{ma}^7$; Am7, Dm7, G7, $\mathrm{C}\textsc{ma}^7$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{Am7}}{\mathrm{VI}^C \vdash \mathrm{VI}^C} \quad \dfrac{\mathrm{Dm7}}{\mathrm{II}^C \vdash \mathrm{II}^C}}{\{\mathrm{VI}, (\mathrm{VI}\backslash \mathrm{II})\}^C \vdash \mathrm{II}^C}\,{\scriptstyle (\backslash_L)} \quad \dfrac{\mathrm{G7}}{\mathrm{V}^C \vdash \mathrm{V}^C}}{\{\mathrm{VI}, \cdots, (\mathrm{II}\backslash \mathrm{V})\}^C \vdash \mathrm{V}^C} \quad \dfrac{\mathrm{C}\textsc{ma}^7}{\mathrm{I}^C \vdash \mathrm{I}^C}}{\{\mathrm{VI}, \cdots, (\mathrm{V}\backslash \mathrm{I})\}^C \vdash \mathrm{I}^C} \quad \dfrac{\vdots}{\{\mathrm{VI}, \cdots, (\mathrm{V}\backslash \mathrm{I})\}^C \vdash \mathrm{I}^C}}{\{\mathrm{VI}, \cdots, (\mathrm{V}\backslash \mathrm{I})\}^C \vdash \mathrm{I}^C}\,{\scriptstyle (\mathrm{CONJ}_2)}$$

This rule is not one of the main rules; it is just a special case of (CONJ). However, we have given it a different name due to the concept of depth, which will be introduced in the next section.

Example 6 (Long distance dependency). In B♭7, G7, E♭MA^7, CMA^7, we encounter a dominant that does not resolve directly to the tonic but instead takes a detour before reaching resolution. Specifically, it moves through the dominant of a tonic that resolves only after passing through several chords. This is a miniature version of what occurs extensively in songs like *Stella by Starlight*, and addressing this type of problem was one of the primary objectives of the method presented here.

$$\dfrac{\dfrac{\dfrac{\dfrac{B\flat 7}{\mathrm{V}^{E\flat} \vdash \mathrm{V}^{E\flat}} \quad \dfrac{G7}{\mathrm{V}^{C} \vdash \mathrm{V}^{C}}}{\mathrm{V}^{E\flat}; \mathrm{V}^{C} \vdash \mathrm{V}^{E\flat} \succ \mathrm{V}^{C}}\,(\succ) \quad \dfrac{\mathrm{E}\flat\mathrm{MA}^7}{\mathrm{I}^{E\flat} \vdash \mathrm{I}^{E\flat}}}{\mathrm{V}^{E\flat}, (\mathrm{V}\backslash\mathrm{I})^{E\flat}; \mathrm{V}^{C} \vdash \mathrm{I}^{E\flat} \succ \mathrm{V}^{C}}\,(;_L) \quad \dfrac{\mathrm{CMA}^7}{\mathrm{I}^{C} \vdash \mathrm{I}^{C}}}{\mathrm{V}^{E\flat}, (\mathrm{V}\backslash\mathrm{I})^{E\flat}; \mathrm{V}^{C}, (\mathrm{V}\backslash\mathrm{I})^{C} \vdash \mathrm{I}^{E\flat} \succ \mathrm{I}^{C}}\,(;_R)$$

The final result can be interpreted as follows: from the dominant in the tonality of E♭ we have a cadence at a certain point and the same happens in the tonality of C, that is connected to E♭. Everything is interpreted in the two tonalities on the right, i.e., E♭ major and C major, that are the two tonalities of the harmonic progression.

Example 7 (Multiple Relations). $A7^{alt}, Dm7, G7, B\flat 7, E\flat\mathrm{MA}^7, C\mathrm{MA}^7$can be analyzed as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A7^{alt}}{\mathrm{V}^{d} \vdash \mathrm{V}^{d}} \quad \dfrac{Dm7}{\mathrm{I}^{d} \vdash \mathrm{I}^{d}}}{\{\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})\}^{d} \vdash \mathrm{I}^{d}}\,(\backslash_L)}{\{\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})\}^{d}, R^{d}_{C} \vdash \mathrm{II}^{C}}\,(R) \quad \dfrac{G7}{\mathrm{V}^{C} \vdash \mathrm{V}^{C}}}{\{\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})\}^{d}, R^{d}_{C}, (\mathrm{II}\backslash\mathrm{V})^{C} \vdash \mathrm{V}^{C}}\,(\backslash_L) \quad \dfrac{\dfrac{B\flat 7}{\mathrm{V}^{E\flat} \vdash \mathrm{V}^{E\flat}} \quad \dfrac{E\flat\mathrm{MA}^7}{\mathrm{I}^{E\flat} \vdash \mathrm{I}^{E\flat}}}{\{\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})\}^{E\flat} \vdash \mathrm{I}^{E\flat}}\,(\backslash_L)}{\{\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})\}^{d}, R^{d}_{C}, (\mathrm{II}\backslash\mathrm{V})^{C}; \{\mathrm{V}, (\mathrm{V}\backslash I)\}^{E\flat} \vdash \mathrm{V}^{C} \succ \mathrm{I}^{E\flat}}\,(\succ) \quad \dfrac{C\mathrm{MA}^7}{\mathrm{I}^{C} \vdash \mathrm{I}^{C}}}{\{\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})\}^{d}, R^{d}_{C}, \{(\mathrm{II}\backslash\mathrm{V}), (\mathrm{V}\backslash\mathrm{I})\}^{C}; \{\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})\}^{E\flat} \vdash \mathrm{I}^{E\flat} \succ \mathrm{I}^{C}}\,(;_L)$$

This example represents a synthesis of all previous examples. We begin with the dominant of D minor, which resolves. However, we also observe a relation to the tonality of C major, as we encounter the dominant G7. Thus, we establish an accessibility relation between D minor and C major, transforming the function of D minor from a tonic into a subdominant (labelled with a second degree) within a cadence, which remains for now unresolved.

Following the G7 chord, a different cadence appears in E♭, moving from the dominant to the tonic. This creates a new accessibility relation, R, which does not alter the functional role of the preceding chords. The resolution of the original cadence is deferred until after the cadence in E♭. The tonalities of this cadence will be E♭ major after C major.

4 The depth of an Harmonic Analysis

It is insightful to use the logical notion of the depth of a proof to track the complexity of a Harmonic Analysis (cfr. [20]). The idea is that if a large number of rules are applied during the analysis of a harmonic structure, the complexity of that structure increases. This measure of complexity is objective and not based on aesthetics; rather, it provides a systematic way of quantifying harmonic intricacy. Hereafter, Let $\mathfrak{a}, \mathfrak{b}, \ldots$ stand for sequents.

Definition 7 (Depth of an Harmonic Analysis). *The Depth of an harmonic set of chords* $\mathfrak{a}$*,* $dp(\mathfrak{a})$*, is defined as follows:*

$$
\begin{aligned}
&dp(x^\alpha \vdash x^\alpha) = 0\\
&dp(\mathbb{A} \vdash (x\backslash y)^\alpha) = dp(x^\alpha, \mathbb{A} \vdash y^\alpha)\\
&dp(\mathbb{B}, \mathbb{A}, (x\backslash y)^\alpha, \mathbb{C} \vdash z^\beta) = dp(\mathbb{A} \vdash x^\alpha) + dp(\mathbb{B}, y^\alpha, \mathbb{C} \vdash z^\beta) + 1\\
&dp(\mathbb{A}, R^\alpha_\beta \vdash y^\beta) = dp(\mathbb{A} \vdash x^\alpha)\\
&dp(\mathbb{A}, \mathbb{B} \vdash x^\alpha) = dp(\mathbb{A} \vdash x^\alpha) + dp(\mathbb{B} \vdash x^\alpha)\\
&dp(\mathbb{A}; \mathbb{B} \vdash x^\alpha \succ z^\beta) = dp(\mathbb{A} \vdash x^\alpha) + dp(\mathbb{B} \vdash z^\beta) + 1\\
&dp(\mathbb{A}, (x\backslash z)^\alpha; \mathbb{B} \vdash z^\alpha \succ y^\beta) = dp(\mathbb{A}; \mathbb{B} \vdash x^\alpha \succ y^\beta) + dp(z^\alpha \vdash z^\alpha) + 1\\
&dp(\mathbb{A}; \mathbb{B}, (y\backslash z)^\beta \vdash x^\alpha \succ z^\beta) = dp(\mathbb{A}; \mathbb{B} \vdash x^\alpha \succ y^\beta) + dp(z^\beta \vdash z^\beta) + 1
\end{aligned}
$$

The depth of a Harmonic Analysis in LLCM is increased only by the rules $(\backslash_L)$ and $(\succ)$, but not by the R rule. This is due to the fact that when there is a progression from a chord to the same chord, we do not want to increase the complexity of the harmonic structure. Furthermore, when the harmonic analysis is CONJ-ed, the depth of the harmonic set of chords will correspond to that of only one of the two branches of the analysis.

Remark 1 (The depth of ($CONJ_2$)). The depth, according to the rules we have written, can only increase. The only case where it does not increase is when ($CONJ_2$) is used. In this case, we have a contraction due to the repetition of a certain set of chords, so the depth remains the same as one of the repetitions.

Example 8. Let us consider the analysis in Example 3, the depth of the Harmonic Analysis will be 3, because we have applied only 3 rules to the derivation.

Definition 8 (Decorated turnstile). *To track the evolution of the depth in the Harmonic Analysis, we can decorate the turnstiles with a number that increases according to the rules. The general form is as follows:*

$$\mathbb{A} \vdash_{dp(\mathbb{A} \vdash x^\alpha)} x^\alpha$$

The rules introduced in Section 3 can then be rewritten with decorated sequents indicating the depth of the Harmonic Analysis, as shown in Appendix A.

Example 9. For the chord progression B♭7, E♭MA^7, G7, CMA^7:

$$\dfrac{\dfrac{\dfrac{B\flat 7}{\mathrm{V}^{E\flat} \vdash_0 \mathrm{V}^{E\flat}} \qquad \dfrac{\mathrm{E\flat MA}^7}{\mathrm{I}^{E\flat} \vdash_0 \mathrm{I}^{E\flat}}}{\{\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})\}^{E\flat} \vdash_1 \mathrm{I}^{E\flat}}\,(\backslash_L) \qquad \dfrac{\dfrac{G7}{\mathrm{V}^{C} \vdash_0 \mathrm{V}^{C}} \qquad \dfrac{\mathrm{CMA}^7}{\mathrm{I}^{C} \vdash_0 \mathrm{I}^{C}}}{\{\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})\}^{C} \vdash_1 \mathrm{I}^{C}}\,(\backslash_L)}{\{\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})\}^{E\flat}; \{\mathrm{V}, (\mathrm{V}\backslash\mathrm{I})\}^{C} \vdash_3 \mathrm{I}^{E\flat} \succ \mathrm{I}^{C}}\,(\succ)$$

Example 10 (Different trees, different analysis, same depth). This example introduces an intriguing ambiguity in harmonic analysis, specifically regarding the dual interpretation of the D7 chord in *Stella by Starlight*. On one hand, D7 can be understood as the dominant of G, adhering to the conventional dominant-to-tonic progression, while on the other hand, D7 may be seen as the tonic in a cadence that begins with A7, where A7 functions as its dominant. Despite these distinct interpretive paths, both analyses apply the same number of rules and arrive at the same final sequent, demonstrating the system's ability to accommodate varying harmonic perspectives while maintaining structural consistency.

In the first example, we initially consider D7 as the dominant of G7$^{\sharp 5}$. Before this, we take it into account with Am7$^{(\flat 5)}$, and after applying the (R) rule, we observe how the interpretation changes on the right, allowing the application of the $(\backslash_L)$ rule. Then, a new application of the (R) rule is required to complete the analysis.

$$\dfrac{\dfrac{\dfrac{\dfrac{A7}{\mathrm{V}^{D} \vdash_0 \mathrm{V}^{D}} \qquad \dfrac{\dfrac{\dfrac{Am7^{(\flat 5)}}{\mathrm{II}^{G} \vdash_0 \mathrm{II}^{G}} \qquad \dfrac{D7}{\mathrm{V}^{G} \vdash_0 \mathrm{V}^{G}}}{\mathrm{II}^{G}, (\mathrm{II}\backslash\mathrm{V})^{G} \vdash_1 \mathrm{V}^{G}}\,(\backslash_L)}{\mathrm{II}^{G}, (\mathrm{II}\backslash\mathrm{V})^{G}, R^{G}_{D} \vdash_1 \mathrm{I}^{D}}\,(R)}{\mathrm{V}^{D}, R^{D}_{G}, \mathrm{II}^{G}, (\mathrm{II}\backslash\mathrm{V})^{G}, R^{G}_{D}, (\mathrm{V}\backslash\mathrm{I})^{D} \vdash_2 \mathrm{I}^{D}}\,(\backslash_L)}{\mathrm{V}^{D}, R^{D}_{G}, \mathrm{II}^{G}, (\mathrm{II}\backslash\mathrm{V})^{G}, R^{G}_{D}, (\mathrm{V}\backslash\mathrm{I})^{D}, R^{D}_{G} \vdash_2 \mathrm{V}^{G}}\,(R) \qquad \dfrac{G7^{\sharp 5}}{\mathrm{I}^{G} \vdash_0 \mathrm{I}^{G}}}{\mathrm{V}^{D}, R^{D}_{G}, \mathrm{II}^{G}, (\mathrm{II}\backslash\mathrm{V})^{G}, R^{G}_{D}, (\mathrm{V}\backslash\mathrm{I})^{D}, R^{D}_{G}, (\mathrm{V}\backslash I)^{G} \vdash_3 \mathrm{I}^{G}}\,(\backslash_L)$$

This tree emphasizes that in these four chords there is a turnaround that keeps us in the tonality of G, as it is possible to read on the right part of the sequent. However, LLCM let us to stress different properties of the chords and this is the case of the following trees.

In the example 4.3B, the D7 is once again interpreted as the dominant of G, but this time we evaluate the first two chords separately, followed by the rest of the harmonic structure. The (R) rule is not applied in this case.

$$\dfrac{\dfrac{\dfrac{\dfrac{A7}{\mathrm{V}^{D} \vdash_0 \mathrm{V}^{D}} \qquad \dfrac{Am7^{(\flat 5)}}{\mathrm{II}^{G} \vdash_0 \mathrm{II}^{G}}}{\mathrm{V}^{D}; \mathrm{II}^{G} \vdash_1 \mathrm{V}^{D} \succ \mathrm{II}^{G}}\,(\succ) \qquad \dfrac{D7}{\mathrm{V}^{G} \vdash_0 \mathrm{V}^{G}}}{\mathrm{V}^{D}; \mathrm{II}^{G}, (\mathrm{II}\backslash\mathrm{V})^{G} \vdash_2 \mathrm{V}^{D} \succ \mathrm{V}^{G}}\,(;_R) \qquad \dfrac{G7^{\sharp 5}}{\mathrm{I}^{G} \vdash_0 \mathrm{I}^{G}}}{\mathrm{V}^{D}; \mathrm{II}^{G}, (\mathrm{II}\backslash\mathrm{V})^{G}, (\mathrm{V}\backslash\mathrm{I})^{G} \vdash_3 \mathrm{V}^{D} \succ \mathrm{I}^{G}}\,(\backslash_L)$$

Here, the analysis emphasizes that the chords lead us to the tonality of G, but only after an unresolved suspended cadence on D.

In the example 4.3C, we initially interpret D7 as the tonic of a cadence on D. We then embed the D7 to the half cadence of V^D, and can make the first half compose a perfect cadence.

$$\dfrac{\dfrac{\dfrac{\dfrac{A7}{\mathrm{V}^D \vdash_0 \mathrm{V}^D} \quad \dfrac{Am7^{(\flat 5)}}{\mathrm{II}^G \vdash_0 \mathrm{II}^G}}{\mathrm{V}^D; \mathrm{II}^G \vdash_1 \mathrm{V}^D \succ \mathrm{II}^G}\,(\succ) \quad \dfrac{D7}{\mathrm{I}^D \vdash_0 \mathrm{I}^D}}{\mathrm{V}^D, (\mathrm{V}\backslash\mathrm{I})^D; \mathrm{II}^G \vdash_2 \mathrm{II}^G \succ \mathrm{I}^D}\,(;_L) \quad \dfrac{G7^{\sharp 5}}{\mathrm{I}^G \vdash_0 \mathrm{I}^G}}{\mathrm{V}^D, (\mathrm{V}\backslash\mathrm{I})^D; \mathrm{II}^G, (\mathrm{II}\backslash\mathrm{I})^G \vdash_3 \mathrm{I}^D \succ \mathrm{I}^G}\,(\backslash_L)$$

This tree highlights that there are two distinct cadences, one on D and one on G; both are resolved but, in a sense, remain separate. Although in this example, 4.3A is the most convincing, all the trees may be valuable to the analyst, as LLCM allows for different interpretations while maintaining the same depth.

The stability of the system can be seen both on the final sequent and on the final value of the depth of the Harmonic Analysis. It is possible to generalize over the idea that the different analysis, also if they point out different characteristics, they have the same depth and obtain the following theorem.

Theorem 1 (Depth uniqueness of Harmonic Analysis). *Given a sequence of chords the number of rules applied to obtain a terminating derivation is the same for every possible derivation, i.e., every harmonic analysis has the same depth.*

Proof. Proof by induction on the depth of the analysis; the base case, where only one rule is applied, is straightforward by definitions. Then, consider a formula $\Gamma \vdash \Delta$ that involves n applications of rules. We need to prove the statement for the case when there are $n+1$ rules. By definition, the depth $n+1$ can only be reached through the application of the rules $(\backslash_L)$ and $(\succ)$ by definition. Therefore, it follows that the depth $n+1$ must necessarily be reached by an application of one of these rules, and no other rules. This means that the final depth of the analysis will remain fixed, even if the analysis is carried out in a different order. □

For a systematic example of the application of the depth of the harmonic analysis see Appendix B, where *In your own sweet way* is analysed. The harmonic structure is interesting in various way: it has changes in function, long distant dependencies and a lot of tricky passages.

5 Conclusions

In this paper, we have introduced the Labelled Lambek Calculus for Music analysis (LLCM), as an extension of traditional Lambek Calculus designed specifically to address the complexities of harmonic progressions in tonal music. Our system brings several significant advancements to the formal analysis of music, which we summarize as follows:

- **Formalization of Harmonic Analysis:** LLCM provides a rigorous and structured approach for analyzing harmonic sequences, improving upon traditional methods by introducing explicit rules that govern tonal shifts, cadences, and other musical phenomena. The use of labelled sequent calculus allows for a clearer representation of relationships between different tonalities, enhancing the formal treatment of music analysis.
- **Interconnected Tonalities and Accessibility Relations:** The introduction of accessibility relations in LLCM allows for the connection of different tonalities within a harmonic progression. This feature is crucial for analyzing music that involves multiple tonal centers or complex modulations, as it captures the interconnectedness of distinct tonal regions and enables a coherent representation of long-distance dependencies.
- **Depth as a Measure of Complexity:** We introduced a metric for the *depth* of a harmonic analysis, which quantifies the structural intricacy of a harmonic sequence. This depth measure, based on the number of rules applied during the analysis, provides an objective way to compare different harmonic structures and evaluate their complexity.
- **Handling Ambiguities in Interpretation:** LLCM is designed to be flexible in accommodating various interpretations of the same harmonic sequence. Different analyses can be performed using different interpretative paths, yet they result in the same final depth and harmonic structure, demonstrating the system's ability to handle ambiguities while maintaining analytical consistency.
- **Application to Real-World Music Examples:** The system's practical utility was demonstrated through the analysis of well-known pieces, such as *All the Things You Are* and *In your own sweet way.* These examples illustrate LLCM's capacity to analyze complex harmonic progressions, including long-distance dependencies and multi-tonal structures, confirming its effectiveness as a tool for formal music analysis.

In conclusion, LLCM represents a significant advancement in the formal proof-theoretic approach to music analysis, offering a systematic and quantitative framework for evaluating the complexity of harmonic progressions.

5.1 Future Work

There are numerous avenues for further exploration and development:

- **Investigating the Accessible Relation R:** A deeper exploration of the concept of accessible relation R in music could yield insights into how different musical elements interact within various contexts. This could involve analyzing how R can be defined and utilized across different genres and styles.
- **Applying LLCM Principles to Composition:** Rather than limiting the application of LLCM to analysis, reversing the perspective to apply similar principles in the composition of harmonic structures could be valuable. This could involve developing guidelines or frameworks that composers can use to create coherent and innovative harmonic progressions based on the established rules of LLCM.
- **Extending LLCM to Modal Music:** There is potential for creating an analogous system tailored for modal music. This would involve adapting the principles of LLCM to accommodate the unique characteristics of modal scales and harmonic relationships, enabling a comprehensive analysis of modal compositions. This gerarchic, tree-based system can be applied to modal harmony, but also to modal chants without harmony, like Gregorian one, due to the gerarchy that it is possible to find in them (*finalis*, *repercussio*. . .).
- **Developing a Computational Version of LLCM:** Creating a computational model of this system would enable automated analysis of a wide variety of harmonic structures. Such a tool could facilitate musicologists, educators, and composers in examining complex harmonic relationships quickly, providing valuable insights into both traditional and contemporary works.
- **Adjusting LLCM for Genre and Era Classification:** Modifying the system to effectively classify songs into specific genres and eras would enhance its practical applications. This could involve identifying key characteristics associated with different musical styles and periods, allowing for a more nuanced understanding of how harmonic structures evolve over time and across various musical traditions.
- **Introduce** $(\succ_L)$ **and** $(\succ_R)$: In this paper, we have used $(;_R)$ and $(;_L)$. In the future, it may be interesting to make LLCM more similar to Lambek Calculus by introducing two different rules for $(\succ)$, one on the right and one on the left. Since we have not yet found a semantics, we did not employ it in the paper, but it may be possible to do so in the future to improve logical harmony and clarity.

A Rules with depth

Rules of LLCM with the decorated sequents

Initial Sequent:

$$\frac{\text{Chord}}{x^\alpha \vdash_0 x^\alpha}$$

Depth of the trees:

$$\begin{aligned}
dp(\mathfrak{a}) &= dp(\mathbb{A}, x^\alpha, \mathbb{B} \vdash z^\beta)\\
dp(\mathfrak{b}) &= dp(\mathbb{A} \vdash x^\alpha)\\
dp(\mathfrak{c}) &= dp(y^\alpha, \mathbb{B} \vdash z^\beta)\\
dp(\mathfrak{d}) &= dp(\mathbb{A}, x^\alpha, \mathbb{B} \vdash z^\beta)\\
dp(\mathfrak{e}) &= dp(\mathbb{B} \vdash z^\beta)\\
dp(\mathfrak{f}) &= dp(\mathbb{A}, x^\alpha \vdash x^\alpha)\\
dp(\mathfrak{g}) &= dp(y^\beta, \mathbb{B} \vdash x^\alpha)
\end{aligned}$$

Rules:

$$\frac{\mathbb{A}, x^\alpha, \mathbb{B} \vdash_{\mathrm{dp}(\mathfrak{a})} y^\alpha \succ z^\beta}{\mathbb{A}, (x\backslash y)^\alpha, \mathbb{B} \vdash_{\mathrm{dp}(\mathfrak{a})} z^\beta}\ (\backslash_R)$$

$$\frac{\mathbb{A} \vdash_{\mathrm{dp}(\mathfrak{b})} x^\alpha \qquad y^\alpha, \mathbb{B} \vdash_{\mathrm{dp}(\mathfrak{c})} z^\beta}{\mathbb{A}, (x\backslash y)^\alpha, \mathbb{B} \vdash_{\mathrm{dp}(\mathfrak{b})+\mathrm{dp}(\mathfrak{c})+1} z^\beta}\ (\backslash_L)$$

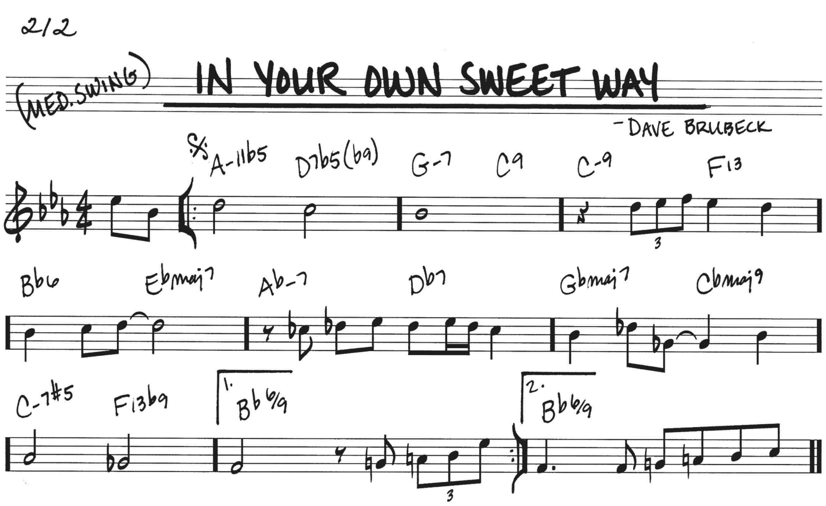
$$\frac{\mathbb{A}, x^\alpha, \vdash_{\mathrm{dp}(\mathfrak{d})} z^\beta}{\mathbb{A}, x^\alpha, R^\alpha_\beta \vdash_{\mathrm{dp}(\mathfrak{d})} z^\beta}\ (R)$$

$$\frac{\mathbb{A} \vdash_{\mathrm{dp}(\mathfrak{d})} x^\alpha \qquad \mathbb{B} \vdash_{\mathrm{dp}(\mathfrak{e})} z^\beta}{\mathbb{A}; \mathbb{B} \vdash_{\mathrm{dp}(\mathfrak{d})+\mathrm{dp}(\mathfrak{e})+1} x^\alpha \succ z^\beta}\ (\succ)$$

$$\frac{\mathbb{A} \vdash_{\mathrm{dp}(\mathfrak{b})} x^\alpha \qquad \mathbb{B} \vdash_{\mathrm{dp}(\mathfrak{g})} x^\alpha}{\mathbb{A}, \mathbb{B} \vdash_{\mathrm{dp}(\mathfrak{b})+\mathrm{dp}(\mathfrak{g})} x^\alpha}\ (\mathrm{CONJ})$$

B Analysis example

Analysis of the first 8 bars (A) of *In your own sweet way* by Dave Brubeck.



$$\boxed{\mathrm{A}}: \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Am7^{(\flat 5)}}{\mathrm{II}^g \vdash_0 \mathrm{II}^g} \quad \dfrac{D7^{(\flat 5)}}{\mathrm{V}^g \vdash_0 \mathrm{V}^g}}{\mathrm{II}^g, (\mathrm{II}\backslash\mathrm{V})^g \vdash_1 \mathrm{V}^g}\,(\backslash_L) \quad \dfrac{Gm7}{\mathrm{I}^g \vdash_0 \mathrm{I}^g}}{\mathrm{II}^g, (\mathrm{II}\backslash\mathrm{V})^g, (\mathrm{V}\backslash\mathrm{I})^g \vdash_2 \mathrm{I}^g}\,(\backslash_L)}{\{\mathrm{II}, (\mathrm{II}\backslash\mathrm{V}), (\mathrm{V}\backslash\mathrm{I})\}^g, R_F^g \vdash_2 \mathrm{II}^F}\,(R) \quad \dfrac{C7}{\mathrm{V}^F \vdash_0 \mathrm{V}^F}}{\{\mathrm{II}, (\mathrm{II}\backslash\mathrm{V}), (\mathrm{V}\backslash\mathrm{I})\}^g, R_F^g, (\mathrm{II}\backslash\mathrm{V})^F \vdash_3 \mathrm{V}^F}\,(\backslash_L)$$

$$\boxed{\mathrm{B}}: \quad \dfrac{\dfrac{\dfrac{\boxed{\mathrm{A}} \quad \dfrac{\dfrac{\dfrac{Cm7}{\mathrm{II}^{B\flat} \vdash_0 \mathrm{II}^{B\flat}} \quad \dfrac{F7}{\mathrm{V}^{B\flat} \vdash_0 \mathrm{V}^{B\flat}}}{\{\mathrm{II}, (\mathrm{II}\backslash\mathrm{V})\}^{B\flat} \vdash_1 \mathrm{V}^{B\flat}}\,(\backslash_L)}{\{\mathrm{II}, (\mathrm{II}\backslash\mathrm{V})\}^{B\flat}, R_F^{B\flat} \vdash_1 \mathrm{I}^F}\,(R)}{\cdots, R_F^g, \{\mathrm{II}, (\mathrm{II}\backslash\mathrm{V})\}^{B\flat}, R_F^{B\flat}, (\mathrm{V}\backslash\mathrm{I})^F \vdash_5 \mathrm{I}^F}\,(\backslash_L)}{\cdots, R_F^{B\flat}, (\mathrm{V}\backslash\mathrm{I})^F, R_{B\flat}^F \vdash_5 \mathrm{V}^{B\flat}}\,(R) \quad \dfrac{B\flat\mathrm{MA}^7}{\mathrm{I}^{B\flat} \vdash_0 \mathrm{I}^{B\flat}}}{\{\mathrm{II}, (\mathrm{II}\backslash\mathrm{V}), (\mathrm{V}\backslash\mathrm{I})\}^g, R_F^g, (\mathrm{II}\backslash\mathrm{V})^F, R_{B\flat}^F, \cdots, R_{B\flat}^F, (\mathrm{V}\backslash\mathrm{I})^{B\flat} \vdash_6 \mathrm{I}^{B\flat}}\,(\backslash_L)$$

$$\boxed{\mathrm{C}}: \quad \dfrac{\dfrac{\boxed{\mathrm{B}} \quad \dfrac{E\flat\mathrm{MA}^7}{\mathrm{IV}^{B\flat} \vdash_0 \mathrm{IV}^{B\flat}}}{\{\mathrm{II}, (\mathrm{II}\backslash\mathrm{V}), (\mathrm{V}\backslash\mathrm{I})\}^g, R_F^g, \cdots, R_{B\flat}^F, \{(\mathrm{V}\backslash\mathrm{I}) \cdot (\mathrm{I}\backslash\mathrm{IV})\}^{B\flat} \vdash_7 \mathrm{IV}^{B\flat}}\,(\backslash_L)}{\mathrm{II}^g, \cdots, (\mathrm{V}\backslash\mathrm{I})^{B\flat}, (\mathrm{I}\backslash\mathrm{IV})^{B\flat} \vdash_7 \mathrm{IV}^{B\flat}}$$

$$\boxed{\mathrm{D}}: \quad \dfrac{\dfrac{\dfrac{\dfrac{A\flat m7}{\mathrm{II}^{G\flat} \vdash_0 \mathrm{II}^{G\flat}} \quad \dfrac{D\flat 7}{\mathrm{V}^{G\flat} \vdash_0 \mathrm{V}^{G\flat}}}{(\mathrm{II}\backslash\mathrm{V})^{G\flat} \vdash_1 \mathrm{V}^{G\flat}}\,(\backslash_L) \quad \dfrac{G\flat\mathrm{MA}^7}{\mathrm{I}^{G\flat} \vdash_0 \mathrm{I}^{G\flat}}}{\{(\mathrm{II}\backslash\mathrm{V}), (\mathrm{V}\backslash\mathrm{I})\}^{G\flat} \vdash_2 \mathrm{I}^{G\flat}} \quad \dfrac{C\flat\mathrm{MA}^7}{\mathrm{IV}^{G\flat} \vdash_3 \mathrm{IV}^{G\flat}}}{\{(\mathrm{II}\backslash\mathrm{V}), (\mathrm{V}\backslash\mathrm{I}), (\mathrm{I}\backslash\mathrm{IV})\}^{G\flat} \vdash_6 \mathrm{IV}^{G\flat}}\,(\backslash_L)$$

$$\boxed{\mathrm{E}}: \quad \dfrac{\boxed{\mathrm{C}} \quad \boxed{\mathrm{D}}}{\mathrm{II}^g, \cdots, \{(\mathrm{V}\backslash\mathrm{I}), (\mathrm{I}\backslash\mathrm{IV})\}^{B\flat}; \{(\mathrm{II}\backslash\mathrm{V}), (\mathrm{V}\backslash\mathrm{I}), (\mathrm{I}\backslash\mathrm{IV})\}^{G\flat} \vdash_{14} \mathrm{IV}^{B\flat} \succ \mathrm{IV}^{G\flat}}\,(\succ_R)$$

$$\dfrac{\boxed{\mathrm{E}} \quad \dfrac{\dfrac{\dfrac{Cm7^{\sharp 5}}{\mathrm{II}^{B\flat} \vdash_0 \mathrm{II}^{B\flat}} \quad \dfrac{F7}{\mathrm{V}^{B\flat} \vdash_0 \mathrm{V}^{B\flat}}}{\{\mathrm{II}, (\mathrm{II}\backslash\mathrm{V})\}^{B\flat} \vdash_1 \mathrm{V}^{B\flat}}\,(\backslash_L) \quad \dfrac{B\flat\mathrm{MA}^7}{\mathrm{I}^{B\flat} \vdash_0 \mathrm{I}^{B\flat}}}{\{\mathrm{II}, (\mathrm{II}\backslash\mathrm{V}), (\mathrm{V}\backslash\mathrm{I})\}^{B\flat} \vdash_2 \mathrm{I}^{B\flat}}\,(\backslash_L)}{\mathrm{II}^g, \cdots, \{\mathrm{II}, (\mathrm{II}\backslash\mathrm{V}), (\mathrm{V}\backslash\mathrm{I})\}^{B\flat} \vdash_{17} \mathrm{IV}^{B\flat} \succ \mathrm{IV}^{G\flat} \succ \mathrm{I}^{B\flat}}\,(\succ_R)$$

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