Belief Structures within Fractional Semantics: an overview

1.1 Introduction

Fractional Semantics, initially introduced in [17], serves as a powerful tool for discerning the number of proper axioms within a proposition relative to the total number of axioms. This method underwent refinement for modal logic [19] and expanded into the domain of beliefs in [3] and applied to the Lottery Paradox in [2]. The study demonstrated the instrumental role of Fractional Semantics in resolving the Lottery Paradox.

This work has two main objectives: firstly, to present in a refined way $GS4_B$, firstly presented in [3]—the Fractional Semantics System that incorporates beliefs; secondly, to introduce a nuanced categorization of beliefs. In [3], all beliefs are treated as true, akin to tautologies. However, this poses a philosophical challenge, as not every proposition we believe aligns with the certainty of a tautology. To address this, we utilize Hyperreal numbers, signifying that a belief holds a value not of 1, but infinitesimally lower—specifically, $1-\delta$, where δ represents an infinitesimal value smaller than every real number.

This approach draws inspiration from Hansson [9, 10], who used hyperreal numbers to differentiate between Full Beliefs (assigned a value of 1) and beliefs open to revision in the presence of evidence, termed Revisable Beliefs. However, our aim is different: we seek a system capable of tracking not only the count of Full Beliefs but also beliefs considered true even if subject to revision, differenciating between them thanks to hyperreal numbers. Fractional Semantics enables us to perform derivations and determine the composition of the combination between tautologies, Full Beliefs, and Revisable Beliefs.

The paper is structured as follows: in the first section, we briefly present Fractional Semantics, referring to [2, 3, 17–19] for more examples and proofs; in the second section, we present proofs for theorems from [3]; and in the last section, we introduce a distinction between Full Beliefs and Revisable Beliefs within the framework of fractional semantics.

1.2 Fractional Semantics

Fractional semantics is a multi-valued approach governed by pure proof-theoretic considerations firstly introduced in [17], assigning truth values as rational numbers in the closed interval [0,1] breaking the symmetry between tautologies and contradictions, allowing values other than 0 for non-logical axioms, i.e., contingent. It measures the proposition's proximity to being a tautology or a contradiction.

To enable fractional interpretation, a decidable logic $\mathcal{L}$ is required, displayed in a sequent system **S** meeting three conditions: bilateralism, invertibility, and stability.

Bilateralism : **S**, as a bilateral system, generates **S**-derivations for any well formed formula *A* of $\mathcal{L}$: if *A* is valid, its **S**-derivation will be an actual proof of *A*; if *A* is invalid, its **S**-derivation will provide a formal refutation of *A*, i.e., a proof of its unprovability.

Invertibility : each logical rule of *S* is invertible, meaning that the provability of its conclusion implies the provability of (each one of) its premise(s). This means that there is an algorithm to decompose uniquely a sequent into an equivalent formula in conjunctive normal form.

Stability : two analytic S-proofs with the same end-sequent share the same multi-set of top-sequents.

Fractional semantics is obtained by focusing on the axiomatic structure of proofs expressed in Kleene's one-side sequent system $GS4$ [13, 22]. The system is as following:

$$\frac{}{\vdash \Gamma, p, \overline{p}}\ (ax.)$$

$$\frac{\vdash \Gamma, p, q}{\vdash \Gamma, p \vee q}\ (\vee) \qquad\qquad \frac{\vdash \Gamma, p \qquad \vdash \Gamma, q}{\vdash \Gamma, p \wedge q}\ (\wedge)$$

GS4 is a one-sided sequent where structural properties are absorbed into the calculus, Γ and Δ are multisets of formulas, and $p, q, \ldots$ are atomic formulas. As usual, $\wedge$ indicates the conjunction and $\vee$ the disjunction. There is not a rule governing negation as it is inductively defined by different atomic formulas p and $\overline{p}$, where $\overline{p}$ indicates the negation of p. Sequents can be decomposed into initial sequents that are allowed to contain only atomic formulas.

The interpretation of a formula is the result of the ratio between the number of identity top-sequents $(\Delta, p, \overline{p})$ out of the total number of top-sequents occurring in any of its proofs. Weakening and contraction are dropped while cut rule has the form:

$$\frac{\vdash \Gamma, p \qquad \vdash \overline{p}, \Delta}{\vdash \Gamma, \Delta} \; (cut)$$

In order to give a fractional interpretation a counterpart is needed, namely $\overline{\overline{GS4}}$, that is the $GS4$ calculus maximally extended:

Definition 1.1 ($\overline{\overline{GS4}}$)**.** The calculus $\overline{\overline{GS4}}$ is obtained from $GS4$ that is able to prove any sequent and it satisfies cut-elimination à la Gentzen if its axioms introduce only clauses [17], i.e., a sequent which consists solely of atomic formulae [1].

Definition 1.2 (Top-sequents axioms)**.**

$top^1(\pi)$: denotes the multiset of all and only π's *top-sequents* introduced by an identity axiom, i.e., those those sequents directly introduced as instances of the axiom rules.;

$top^0(\pi)$: denotes the multiset of all and only π's *top-sequents* introduced by a complementary axiom, in other words, those axioms that are not tautological.

Any formula A can be interpreted as the ratio between the number of identity top-sequents (sequents introduced by the standard axiom) out of the total number of top-sequents.

$$[\![A]\!] = \frac{top^1(\pi)}{top^1(\pi) + top^0(\pi)}$$

Definition 1.3 (Top-sequents)**.** Top-sequents represent the number of the leaves of the proof as defined in Definition 1.2 and $[\![\Gamma]\!]$ denotes the value of the formula $\vee\Gamma$ where only $\vee-$applications appear.

$top^1(\pi)$: let's call this *m*

$top^0(\pi)$: let's call this *n*

$[\![\vee\Gamma]\!]$ is $\frac{m}{n} \in [0, 1]$

From this definition it is possible to give general rules with decorated sequents. These decorated sequents are able to keep track of the fractional value along the proof.

$$\frac{}{\vdash^{1}_{1} \Gamma, p, \overline{p}}\ (ax.) \qquad \frac{}{\vdash^{0}_{1} \Delta}\ (\overline{ax.})$$

$$\frac{\vdash^{m}_{n} \Gamma, A, B}{\vdash^{m}_{n} \Gamma, A \vee B}\ (\vee) \qquad \frac{\vdash^{m_1}_{n_1} \Gamma, A \qquad \vdash^{m_2}_{n_2} \Gamma, B}{\vdash^{m_1+m_2}_{n_1+n_2} \Gamma, A \wedge B}\ (\wedge)$$

Example 1.4. Let's consider an example with the turnstile decorated:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} p, q}\ (\overline{ax.})}{\vdash^{0}_{1} p \vee q}\ (\vee) \qquad \dfrac{\dfrac{}{\vdash^{1}_{1} p, \overline{p}}\ (ax.)}{\vdash^{1}_{1} p \vee \overline{p}}\ (\vee)}{\vdash^{1}_{2} (p \vee q) \wedge (p \vee \overline{p})}\ (\wedge) \qquad \dfrac{\dfrac{}{\vdash^{0}_{1} \overline{r}}\ (\overline{ax.}) \qquad \dfrac{}{\vdash^{0}_{1} \overline{t}}\ (\overline{ax.})}{\vdash^{0}_{2} (\overline{r} \wedge \overline{t})}\ (\wedge)}{\vdash^{1}_{4} (p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})}\ (\wedge)$$

Here, it is possible to observe that for each step of the proof, we can directly read the fractional semantics value on the turnstile.

1.2.1 Framing beliefs into Fractional Semantics for classical logic

From Fractional Semantics we can do a different framework where beliefs are incorporated into fractional semantics for classical logic by introducing a set of axioms denoted as B. These axioms, representing the true beliefs of an agent, are treated as tautologies. The underlying philosophy is that an agent naturally considers their own beliefs to be true.

Beliefs in this context are treated as deductively closed, implying that any deduction made using these true beliefs is also considered true. This reflects the idea of an agent being deductively ideal. Integrating such beliefs into fractional semantics can lead to obtaining values greater than those typically permitted by fractional semantics alone.

The inspiration for this expansion comes from one of Makinson's methods, namely *pivotal-assumption consequence*, used to bridge the gap between classical and non-monotonic logic by adding background assumptions. However, the fractional semantics approach with added beliefs differs from *pivotal-assumption consequence* in two key aspects. Firstly, while Makinson used a classical two-valued semantics, fractional semantics operates within a multi-valued interpretation. Secondly, *pivotal-assumption consequence* assigns the value 0 if any axiom is not a proper axiom or belief, whereas fractional semantics can assign values greater than 0 when a top sequent is a tautology or a belief.

To incorporate beliefs into the system, they must be atomic; otherwise, they need to be decomposed. The definitions of $GS4_B$ and $\vdash_B$ are provided as follows:

Definition 1.5 ($GS4_B$)**.** Let $\mathbb{B} = b_1, \ldots, b_n$ a set of non tautological, non contradictory and of arbitrary complexity formulas; let B be the set of sequents obtained from the decomposition of formulas in $\mathbb{B}$ and closed under cut; let $GS4$ be as defined earlier, then $GS4_B$ is the system where everything that is derived from $\mathbb{B}$ and from $GS4$ is true.

Definition 1.6 ($\vdash_B$)**.** If $\vdash$ is the closure relation of classical logic, then $\vdash_B$ is defined as the closure relation of $GS4_B$.

The system is not Post-complete because structurality and consistency are mutually exclusive properties in the axiomatic extension of classical logic: adding new axioms to the system is not possible to mantain structurality, i.e., substitution is dropped. Makinson highlighted this in [15] without explicitly citing Post, even though the underlying reason is identical.

Theorem 1.7. *There is no supra-classical closure relation in the same language as classical* $\vdash$ *that is closed under substitution, except for* $\vdash$ *itself and the total relation i.e. the relation that relates every possible premises to every possible conclusion.*

and this applies also to this system.

Now, let's delve deeper into formalizing the system by defining the top sequent incorporating added beliefs.

Definition 1.8. $top^b(\pi)$: represents the multiset of all and only top sequents of π introduced by a belief.

The reason for introducing this new type of top sequent stems from our desire, particularly in this context, to treat beliefs on par with identity axioms. This is because an agent invariably regards her own beliefs as true. The updated method for calculating the value of a sequent is:

$$[\![A]\!]_B = \frac{top^b(\pi) + top^1(\pi)}{top^b(\pi) + top^1(\pi) + top^0(\pi)}$$

It is also possible to see the same tree with multi valued system, adding a new rule:

$$\frac{}{\,\vdash^{1}_{1\,B} B\,}\ (\overline{b_i})$$

Example 1.9. For example let's consider this example where $B = p, q$

$$
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 \dfrac{
   \dfrac{\dfrac{}{\vdash^1_{1\,B} p,q}\,(b_1)}{\vdash^1_{1\,B} p\lor q}\,(\lor)
   \qquad
   \dfrac{\dfrac{}{\vdash^1_{1\,B} p,\overline{p}}\,(ax.)}{\vdash^1_{1\,B} p\lor \overline{p}}\,(\lor)
 }{\vdash^2_{2\,B} (p\lor q)\land(p\lor\overline{p})}\,(\land)
 \qquad
 \dfrac{\dfrac{}{\vdash^0_{1\,B}\overline{r}}\,(\overline{ax.}) \qquad \dfrac{}{\vdash^0_{1\,B}\overline{t}}\,(\overline{ax.})}{\vdash^0_{2\,B}(\overline{r}\land\overline{t})}\,(\land)
}{\vdash^2_{4\,B} (p\lor q)\land(p\lor\overline{p})\land(\overline{r}\land\overline{t})}\,(\land)
$$

It is worth noting that if this sequent was considered in classical logic, any valuation would assign either the value 0 or 1. Something similar happens in Makinson pivotal assumption consequence, also if the belief set is the same that we have defined earlier, because a two valued logic is there considered.

1.3 Strong cut elimination

The last section pointed out that the agent is an ideal one and that they are aware of every deduction between beliefs. This means that the belief set is deductively closed: nothing that was not already in the set can be derived. In order to have a deductively closed belief set it is important that every combination of sentences, when it is possible, must be closed under cut and the new sentences obtained in this way will be added to the belief set.

In order to eliminate cut from $GS4_B$ the method is taken from [18], but it is simplified because of the nature of one-sided sequents. The method is the following:

1. let's consider a propositional formula $b_i \in B$ (B being the set of beliefs) and decompose it using the invertible rules;

2. the procedures gives identity and non-logical sequents. Remove the identity ones;

3. let's contract every sequent thus obtained;

4. let's consider two sequents Γ, p and $\Delta, \overline{p}$ and add the sequent Γ, Δ to the set of beliefs and let's contract the set thus obtained;

5. the procedure terminates;

6. finally, take the set closed under weakening.

To emphasize the importance of accounting for the fractional value of a formula incorporating beliefs, it is necessary to consider, as initial sequents, not only those obtained directly but also sequents derived via closure under cut. Let's illustrate this with the following example:

Example 1.10. It is easy to show why the step $4.$ is so important. Suppose that an agent has a new belief: $A = (\overline{p} \wedge (\bar{t} \vee q)) \vee (t \wedge (\bar{t} \vee q))$. The first thing to do in order to add that belief is to transform A in a conjunctive form: it is easy to show that it is equivalent to $\vdash (\overline{p} \vee t) \wedge (\bar{t} \vee q) \wedge (t \vee \bar{t} \vee q) \wedge (\bar{t} \vee q)$. Let's decompose it in a set of clauses: $\vdash \overline{p}, t$, $\vdash \bar{t}, q$, $\vdash t, \bar{t}, q$, $\vdash \bar{t}, q$ and remove one of the copies of $\vdash \bar{t}, q$ and the axiom $\vdash t, \bar{t}, q$. By the method presented earlier the agent has to add $(\overline{p} \vee t)$ and $(\bar{t} \vee q)$ to the system, but these beliefs are not cut free. To let them be cut free, it is necessary to close them under the cut.

$$\frac{\vdash \overline{p}, t \qquad \vdash \bar{t}, q}{\vdash \overline{p}, q}\ (cut)$$

From the last point of the method presented earlier, it is needed to add not only $\vdash \overline{p}, t$ and $\vdash \bar{t}, q$, but also $\vdash \overline{p}, q$. Let's see why: $[\![(\overline{p} \vee t) \wedge (\bar{t} \vee q)]\!]$ has value 1 if $B = \{(\overline{p}, t); (\bar{t}, q)\}$

$$\frac{\dfrac{\big|\frac{1}{1}{}_B\ \overline{p}, t}{\big|\frac{1}{1}{}_B\ \overline{p} \vee t}\ (\vee) \qquad \dfrac{\big|\frac{1}{1}{}_B\ \bar{t}, q}{\big|\frac{1}{1}{}_B\ \bar{t} \vee q}\ (\vee)}{\big|\frac{2}{2}{}_B\ (\overline{p} \vee t) \wedge (\bar{t} \vee q)}\ (\wedge)$$

As it was showed, the cut is really important for a complete set of beliefs, but it is also necessary to see how the cut can be eliminated from the calculus.

1.3.1 Elimination of cut

The elimination of cut in presence of proper axioms was firstly proposed by Girard [6], as noted by Avron [1], upgrading the Gentzen's standard cut elimination algorithm. The procedure here proposed, i.e., the decomposition of the formula, the add to the system and the cut of the formula to obtain all the derivations, owes a lot to the one presented in [18].

In the article, in fact, is proved that, for any cluster of extra-logical assumptions, there exists exactly one axiomatic extension of classical propositional logic that admits cut elimination. We can prove that Fractional value does not decrease in $GS4_B$ with relation to the addition of formulas:

Theorem 1.11. *For any multiset of atomic formulas* $\vdash_B \Gamma$ *and* $\vdash_B \Delta$*,* $[\![\bigvee \Gamma \vee \bigvee \Delta]\!]_B \geq [\![\bigvee \Gamma]\!]_B$.

Proof. To prove this is sufficient to consider a transformation of $\vdash_B$. In fact if $B = b_1, \ldots b_n$, then $\vdash_B \Gamma$ is equal to $\vdash \Gamma, \overline{b_1}, \ldots \overline{b_n}$, changing the kind of turnstile from the one introduced here to the classical one, as pointed out in [15][1]. Intuitively this is due to the fact that the sequent is true iff there is a disjunction between a letter and its negation (for example b_i and $\overline{b_i}$). From this fact it is possible to consider four cases:

- if $[\![\Gamma]\!]_B = [\![\Delta]\!]_B = 1$, than obviously $[\![\Gamma \vee \Delta]\!]_B = 1$ as well;
- if $[\![\Gamma]\!]_B = [\![\Gamma, \overline{b_1}, \ldots, \overline{b_n}]\!] = 1$ and $[\![\Delta]\!]_B = 0$, then $[\![\Gamma \vee \Delta]\!]_B = 1$ as well;
- if $[\![\Delta]\!]_B = [\![\Delta, \overline{b_1}, \ldots, \overline{b_n}]\!] = 1$ and $[\![\Gamma]\!]_B = 0$, then $[\![\Gamma \vee \Delta]\!]_B \geq [\![\Gamma]\!]_B$, whatever value assumes $[\![\Gamma]\!]_B$;
- if $[\![\Gamma]\!]_B = [\![\Delta]\!]_B = 0$, then $[\![\Gamma \vee \Delta]\!]_B \geq [\![\Gamma]\!]_B$.

□

It is possible to generalize this result for any context:

Theorem 1.12. *For any context* Γ *and a formula* A*, such that* A *is not contradictory with the set* B*,* $[\![\bigvee \Gamma \vee A]\!]_B \geq [\![\Gamma]\!]_B$.

Proof. Let's prove it by induction on the complexity of the formula A.

Base case: Let's consider A atomic, then we have two cases:

$A \in B$**:** if $A \in B$, then $[\![\bigvee \Gamma, A]\!]_B = 1$ and $[\![\Gamma, A]\!]_B \geq [\![\Gamma]\!]_B$

$A \notin B$**:** if $A \notin B$, then if $[\![\bigvee \Gamma]\!]_B = 1$, there is an atomic formula in Γ that is in the belief set, so also $[\![\bigvee \Gamma, A]\!]_B = 1$. If $[\![\bigvee \Gamma]\!]_B = 0$, $b_1, \ldots, b_n \notin \Gamma$ and then $[\![\bigvee \Gamma \vee A]\!]_B = 0$

Inductive step: Let's consider two cases:

$A \equiv p \wedge q$**:** by inductive hypothesis $[\![\bigvee \Gamma \vee p]\!]_B \geq [\![\Gamma]\!]_B$ and $[\![\bigvee \Gamma \vee q]\!]_B \geq [\![\Gamma]\!]_B$. If at least one between $[\![\bigvee \Gamma \vee p]\!]_B$ and $[\![\bigvee \Gamma \vee q]\!]_B$ is equal to 0, then $[\![\bigvee \Gamma]\!]_B = 0$ for inductive hypothesis and then $[\![\bigvee \Gamma \vee (p \wedge q)]\!]_B \geq [\![\Gamma]\!]_B$. The only remaining case is when $[\![\bigvee \Gamma \vee p]\!]_B = 1$ and $[\![\bigvee \Gamma \vee q]\!]_B = 1$:

[1] In the text the two sided version of this transformation was used, so $\vdash_B \Gamma$ becomes $b_1, \ldots, b_n \vdash \Gamma$, but here because of the choice to use $GS4$ as main system, it is used the one-sided classically equivalent version $\vdash \Gamma, \overline{b_1}, \ldots, \overline{b_n}$.

$$\dfrac{\dfrac{\overline{\big|\!\frac{1}{1}{}_B\ \Gamma, p}}{\big|\!\frac{1}{1}{}_B \bigvee \Gamma \vee p}\,(\vee) \qquad \dfrac{\overline{\big|\!\frac{1}{1}{}_B\ \Gamma, q}}{\big|\!\frac{1}{1}{}_B \bigvee \Gamma \vee q}\,(\vee)}{\big|\!\frac{2}{2}{}_B \bigvee \Gamma \vee (p \wedge q)}\,(\wedge)$$

Thus $[\![\bigvee \Gamma \vee (p \wedge q)]\!]_B \geq [\![\Gamma]\!]_B$.

$A \equiv p \vee q$**:** by theorem 1.11.

□

Theorem 1.13 (Strong cut elimination of $GS4_B$)**.** *The cut rule is redundant when added to $GS4_B$.*

Proof. Girard was the first to notice that a different procedure could preserve cut elimination even in the presence of axioms [6, 16]. The proof is as usual with double induction, the alorithm is similar to the one presented in [20].

□

The set of beliefs can be "completed" through cut or without that. This means that $GS4_B$ is a cut-free system, because it is an axiomatic extension of classical logic. By the way, the use of cut can alter the fractional semantics value as shown in [17]. Thanks to theorem 1.13 the algorithm presented in section 1.3 can be transformed in an algorithm without the presence of cut. As a corollary of the strong cut elimination it can be obtained:

Theorem 1.14 (Uniqueness of axiomatization in $GS4_B$)**.** *For any cluster of axioms in the set of beliefs B the axiomatization is unique.*

Proof. See [18]. □

1.4 Full Beliefs and Revisable Beliefs

The formal model of beliefs introduced since here is a dichotomous system, an all-or-nothing structure, where a belief is either fully accepted or not at all. We have previously asserted that beliefs, within this framework, are deemed true as well as tautologies. However, we can refine this categorization further. In this section, we employ hyperreal numbers, as Hansson did [10], to distinguish between tautologies and beliefs, or more precisely, between Full Beliefs and Revisable Beliefs. These designations are arbitrary and simply signify that a Full Belief is one whose value is immutable, while a Revisable Belief is one that, though currently held as true, remains subject to revision in light of new evidence.

The reason why hyperreal numbers are interesting in this kind of settlement is twofold: on one hand, it is easy to distinguish between Revisable and Full Beliefs; on the other hand, hyperreal numbers do not alter the fractional final value, thereby validating all the proofs that we have made for $GS4_B$ also for this settlement. In fact, $1-\delta$ in $\mathbb{Q}$ is equal to 1, creating a bridge between $GS4_B$ and hyperreal numbers.

A Full Belief is characterized as a belief that remains impervious to revision under any circumstances; it is an assertion that an agent is unwilling to discard in any situation. Conversely, an agent may hold beliefs that are fully accepted, yet subject to revision in light of new evidence; these are termed Revisable Beliefs, in the sense that they are beliefs that, in presence of new evidences, can be revised, while a tautology can be regarded as a Full Belief, because it can't be revised also in presence of new evidences. For the sake of enhancing the generality of the system, we extend this classification beyond tautologies alone. To accomplish this, we adorn the turnstile with the expression $1-\delta$, where δ represents an infinitesimal quantity:

$$\frac{}{\left|\frac{1-\delta}{1}\right._B b_j} \ (b_j)$$

This notation implies that the belief b_j is one of the agent's beliefs, subject to possible revision in a subsequent moment. The symbol δ functions as a label derived during the proof, serving to keep track of the use of one or more propositions that may be revised in the presence of new evidences. This new notation doesn't change the proves of Cut and Weakening Admissibility in function [10]: $st(1-\delta)=1$, because of the fact that for the proofs we can use only the standard part of the hyperreal number. This means that from the point of view of proof theory nothing changed, but something changed in the expressivness of Fractional Semantics. The rule of conjunction still decorate the sequents in the same way:

$$\frac{\left|\frac{m_1}{\mathrm{n}_1}\right. \Gamma, A \qquad \left|\frac{m_2}{\mathrm{n}_2}\right. \Gamma, B}{\left|\frac{m_1+m_2}{\mathrm{n}_1+\mathrm{n}_2}\right. \Gamma, A \wedge B} \ (\wedge)$$

the only difference is that sometimes we will have to add also infinitesimal numbers, for example:

$$\frac{\left|\frac{1-\delta}{1}\right._B p \qquad \left|\frac{1-\gamma}{1}\right._B q}{\left|\frac{2-(\delta+\gamma)}{2}\right._B p \wedge q} \ (\wedge)$$

Also, if the final value seems strange, it indicates that, according to Fractional Semantics, the value remains 1. This implies that the derivation is solely based on true assumptions at the moment of the derivation. On the other hand, we have two infinitesimals, suggesting that two of the assumptions are beliefs that can be discarded in the presence of new evidence. None of the beliefs used are Full Beliefs, so $p \wedge q$ is a proposition with a value of 1, thanks to the set B. The meaning of the value $2 - (\delta + \gamma)$ is that two of the leaves of the tree are beliefs that are possible to revise in the presence of new information. This means that, after revision, the fractional value could also assume a value of 0.5 or maybe also 0 if both of the beliefs once revised result as false. Our idea is that this value is a way to keep track of how many beliefs are not Full beliefs or tautologies into the derivation.

1.4.1 Decomposition of a Revisable Belief

To decompose a belief, the rules remain the same as before; we must decompose it and close under cut. Suppose we aim to incorporate the belief $\vdash q \wedge (r \vee \overline{s})$ into the system, but it is not a full belief. To achieve this, we need to decompose it:

$$\dfrac{\vdash q \qquad \dfrac{\vdash r, \overline{s}}{\vdash r \vee \overline{s}}\,(\vee)}{\vdash q \wedge (r \vee \overline{s})}\,(\wedge)$$

Now, to indicate that the original belief $\vdash q \wedge (r \vee \overline{s})$ was neither a Full Belief nor a Tautology, we adjust its value by adding the number $1 - \delta$ instead of 1. This adjustment accounts for the infinitesimal nature of δ, and its division by 2 ensures the preservation of infinitesimal characteristics.

$$\dfrac{\vdash^{1-\delta}_{1\;B} q \qquad \dfrac{\vdash^{1-\gamma}_{1\;B} r, \overline{s}}{\vdash^{1-\gamma}_{1\;B} r \vee \overline{s}}\,(\vee)}{\vdash^{1-(\delta+\gamma)}_{2\;B} q \wedge (r \vee \overline{s})}\,(\wedge)$$

In the event that either $\vdash q$ or $\vdash r, \overline{s}$ is employed in a derivation, we explicitly denote this value in the sequent derivation. For instance:

$$\dfrac{\dfrac{\dfrac{\vdash^{1}_{1\;B} p, \overline{p}}{\vdash^{1}_{1\;B} p \vee \overline{p}}\,(\vee) \qquad \vdash^{1-\delta}_{1\;B} q}{\vdash^{2-\delta}_{2\;B} (p \vee \overline{p}) \wedge q}\,(\wedge) \qquad \dfrac{\vdash^{0}_{1\;B} \overline{p}, q}{\vdash^{0}_{1\;B} \overline{p} \vee q}\,(\vee)}{\vdash^{2-\delta}_{3\;B} (p \vee \overline{p}) \wedge q \wedge (\overline{p} \vee q)}\,(\wedge)$$

This implies that, even without knowing the initial values of the leaves, we can still make observations about the value $2 - \delta/3$: the standard part of the derivation is the Fractional Semantics value in $GS4_B$ is $2/3$ and we can observe that there is only one infinitesimal number, indicating that only one of the initial beliefs is a Revisable Belief. The portion that is neither a Full Belief nor a Revisable Belief is then $1/3$, representing what remains between $2/3$ and 1.

1.5 Conclusions

The current endeavor to unite Full Beliefs, Revisable Beliefs, and tautologies represents an initial stride towards establishing a connection between Fractional Semantics and Probability. Fractional Semantics emerges as a powerful instrument for delineating the intricacies of a derivation, offering valuable insights into the dynamic evolution of belief values and the interplay between Revisable Beliefs and Full Beliefs throughout the proof. This amalgamation serves as a foundational framework, setting the stage for a more comprehensive exploration of the relationship between Fractional Semantics and Probability.

The forthcoming phase of our research will delve into elucidating the intricate links between Fractional Semantics and Belief Revision. This constitutes another pivotal facet that underscores the significance of the introduced system. The versatility of our system, embracing both the stability of Full Beliefs and the adaptability of Revisable Beliefs, positions it as an invaluable tool for delving into the nuances of belief dynamics and their evolution over the course of iterative revisions. By bridging the gap between Fractional Semantics and Belief Revision, we aim to provide a more holistic understanding of the nuanced interplay between formal semantics and the adaptive nature of belief systems.

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