Framing beliefs into fractional semantics for classical logic

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1 Introduction

The aim of this contribution is to examine one of the potential applications of fractional semantics for classical logic, which was first introduced in [12]. Fractional semantics for classical logic is a multi-valued semantics that is governed by pure proof-theoretic considerations, with truth-values being the rational numbers in the closed interval [0,1]. The primary difference between classical Boolean interpretation and fractional semantics lies in the breaking of symmetry between classical tautologies and contradictions. Fractional semantics can assign values that differ from 0 to both non-logical axioms and contradictions, and can be seen as a way of determining how close a given proposition is to being a tautology or a contradiction.

To allow for a fractional interpretation of its formulas, fractional semantics requires an appropriate proof-theoretic platform, namely a decidable logic $\mathcal{L}$ that can be displayed in a sequent system $\mathbf{S}$ (or its variants) that meets three conditions: bilateralism, invertibility, and stability.

Fractional semantics is obtained by focusing on the axiomatic structure of proofs expressed in Kleene's one-side sequent system $GS4$ [9, 16]. The system has the following rules:

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$$\frac{}{\vdash \Gamma, p, \overline{p}}\ (ax.)$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ (\vee)$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ (\wedge)$$

There is not a rule governing negation as it is inductively defined by different atomic formulas p and $\overline{p}$, where $\overline{p}$ indicates the negation of p. The interpretation of a formula is the result of the ratio between the number of identity top-sequents $(\Delta, p, \overline{p})$ out of the total number of top-sequents occurring in any of its proofs. Weakening and contraction are dropped while cut rule has the form:

$$\frac{\vdash \Gamma, p \qquad \vdash \overline{p}, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

In order to give a fractional interpretation a counterpart is needed, namely $\overline{\overline{GS4}}$, that is the $GS4$ calculus maximally extended:

Definition 1.1 ($\overline{\overline{GS4}}$)**.** $\overline{\overline{GS4}}$ is defined as $GS4$ calculus maximally extended, adding the complementary axiom schema which enables the introduction of whatsoever consistent clause $\vdash \Delta$. The system $\overline{\overline{GS4}}$ is deductively trivial, i.e. anything can be derived in the system.

Definition 1.2 ($[\![\Gamma]\!]$)**.** $[\![\Gamma]\!]$, for each multiset Γ, indicates the fractional semantics value of Γ.

In $\overline{\overline{GS4}}$, for instance, $\vdash p \to (q \wedge p)$ translated in $GS4$ as $\vdash \overline{p} \vee (q \wedge p)$ is derivable:

$$\dfrac{\dfrac{\dfrac{}{\vdash \overline{p}, q}\ \overline{ax.} \qquad \dfrac{}{\vdash \overline{p}, p}\ ax.}{\vdash \overline{p}, q \wedge p}\ (\wedge)}{\vdash \overline{p} \vee (q \wedge p)}\ (\vee)$$

In classical logic the value of this sequent would be 0, but in fractional semantics it is possible to assign a value based on the number of tautological clauses out of two axioms in total, i.e., $[\![\overline{p} \vee (q \wedge p)]\!] = 1/2 = 0.5$, because of the presence of $\vdash \overline{p}, p$. Stability guarantees that any other decomposition of this sequent will always return the value 0.5. It is possible to give a formal definition of top-sequents axioms:

Definition 1.3 (Top-sequents axioms)**.**

$top^1(\pi)$: denotes the multiset of all and only π's *top-sequents* introduced by an identity axiom;

$top^0(\pi)$: denotes the multiset of all and only π's *top-sequents* introduced by a complementary axiom, in other words, those axioms that are not tautological.

Any formula A can be interpreted as the ratio between the number of identity top-sequents (sequents introduced by the standard axiom) out of the total number of top-sequents.

$$[\![A]\!] = \frac{top^1(\pi)}{top^1(\pi) + top^0(\pi)}$$

Example 1.1. Let's take another example to clarify the process. Let's decompose the expression $\vdash (p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash p, q}(\overline{ax.})}{\vdash p \vee q}(\vee) \quad \dfrac{\dfrac{}{\vdash p, \overline{p}}(ax.)}{\vdash p \vee \overline{p}}(\vee)}{\vdash (p \vee q) \wedge (p \vee \overline{p})}(\wedge) \quad \dfrac{\dfrac{}{\vdash \overline{r}}(\overline{ax.}) \quad \dfrac{}{\vdash \overline{t}}(\overline{ax.})}{\vdash (\overline{r} \wedge \overline{t})}(\wedge)}{\vdash (p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})}(\wedge)$$

This proof contains one identity axiom out of four axioms in total and this means that:

$$[\![(p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})]\!] = \frac{1}{4} = 0.25$$

The final value of the sequent A will be 1/4 and it is the same for every decomposition of the considered sequent. Fractional semantics is able to keep trace of the number of contradictions out of the total number of axioms. It is possible to define the fractional semantics in terms of a multi-valued logics, made by this definition:

Definition 1.4 (Top-sequents)**.** Top-sequents represent the number of the leaves of the proof as defined in Definition 1.3 and $[\![\vee\Gamma]\!]$ represents the value assigned to the multiset Γ where only $\vee-$applications appear.

$top^1(\pi)$: let's call this m

$top^0(\pi)$: let's call this n

$[\![\vee\Gamma]\!]$ is $\frac{m}{n} \in [0, 1]$

From this definition it is possible to give general rules:

$$\frac{}{\vdash^{1}_{1} \Gamma, p, \overline{p}} \ (ax.)$$

$$\frac{}{\vdash^{0}_{1} \Delta} \ (\overline{ax.})$$

$$\frac{\vdash^{m}_{\mathrm{n}} \Gamma, A, B}{\vdash^{m}_{\mathrm{n}} \Gamma, A \vee B} \ (\vee)$$

$$\frac{\vdash^{m_1}_{\mathrm{n}_1} \Gamma, A \qquad \vdash^{m_2}_{\mathrm{n}_2} \Gamma, B}{\vdash^{m_1+m_2}_{\mathrm{n}_1+\mathrm{n}_2} \Gamma, A \wedge B} \ (\wedge)$$

This method will permit to keep track of the value at every stage of the proof.

Example 1.2. Let's take the same sequent considered before in example 1.1, but now using this decorated derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} p, q}\,(\overline{ax.})}{\vdash^{0}_{1} p \vee q}\,(\vee) \qquad \dfrac{\dfrac{}{\vdash^{1}_{1} p, \overline{p}}\,(ax.)}{\vdash^{1}_{1} p \vee \overline{p}}\,(\vee)}{\vdash^{1}_{2} (p \vee q) \wedge (p \vee \overline{p})}\,(\wedge) \qquad \dfrac{\dfrac{}{\vdash^{0}_{1} \overline{r}}\,(\overline{ax.}) \qquad \dfrac{}{\vdash^{0}_{1} \overline{t}}\,(\overline{ax.})}{\vdash^{0}_{2} (\overline{r} \wedge \overline{t})}\,(\wedge)}{\vdash^{1}_{4} (p \vee q) \wedge (p \vee \overline{p}) \wedge (\overline{r} \wedge \overline{t})}\,(\wedge)$$

The main difference between Example 1.1 and 1.2 is that in 1.2 is possible to read the fractional semantics value every time a new rule is introduced. For example the fractional semantics value of $\vdash (p \vee q) \wedge (p \vee \overline{p})$ will be $1/2 = 0.5$ as it's easily noticeable in the proof.

2 Framing beliefs into fractional semantics

Now that the general framework is presented, it is possible to see how to consider beliefs in the fractional semantics for classical logic. The idea is simple: a set of axioms, let's say B, that are considered true by an agent, is added to the system and every proposition in it is considered as true as tautologies. The philosophical idea behind this process is that an agent usually considers true their own beliefs.

Beliefs would be considered as deductively closed: this means that every deduction made using true beliefs will be considered true and this also means that the agent is a deductively ideal one. It is interesting to see what happens if the fractional semantics is provided with a system that is able to manage these new axioms, because it will be possible to obtain values greater than the ones that fractional semantics usually permits.

The idea for this kind of expansion was born thanks to the first of the three methods that Makinson used in [11] to bridge the gap between classical and non-monotonic logic, made by adding background assumptions. This kind of method was called *pivotal-assumption consequence* and permitted to infer more than classical logic permits thanks to a set of axioms that are added to the premises in every deduction.

Fractional semantics for classical logic updated with a set of new beliefs is different from pivotal-assumption consequence because of two main reasons. The first one is that Makinson used a classical two-valued semantics, whereas fractional semantics is a multi-valued interpretation. On the one hand *pivotal assumption consequence* would assign the value 0 if at least one of the axioms is neither a proper axiom nor a belief, on the other hand fractional semantics is provided by a system able to assign values greater than 0 when one of the top sequents is a tautology or a belief. The second reason is that, although both of them were born thanks to syntactical techniques, Makinson used an Hilbert-style approach, while fractional semantics uses the Gentzen-style one.

In order to add beliefs to the system, they must be atomic, if they are not, they must be decomposed, as it will be possible to see better.

Definition 2.1 ($GS4_B$)**.** Let $GS4$ as defined earlier, $GS4_B$ is defined as $GS4$ with a set of new axioms, namely $B = b_1, \ldots, b_n$, that represents a non-contradictory set of beliefs of an agent. Each b_i $(1 \leq i \leq n)$ must be atomic.

Definition 2.2 ($\vdash_B$)**.** If $\vdash$ is the closure relation of classical logic, $\vdash_B$ is defined as the closure relation of $GS4_B$.

Makinson [11] pointed out the problems arising when new axioms are added to the system. In fact, substitution is no longer acceptable when the system has the possibility to manage new non-logical axioms. The same result, even if it is not cited by Makinson, is due to the fact that classical logic is a Post-complete system and this means that, once a nontautological formula is added to the system, the new system will be inconsistent, unless structurality is dropped. The system loses the structurality because structurality and consistency are mutually excluding properties in classical logic with extra-logical axioms [13]. In the Makinson's formulation:

Theorem 2.1. *There is no supra-classical closure relation in the same language as classical* $\vdash$ *that is closed under substitution, except for* $\vdash$ *itself and the total relation i.e. the relation that relates every possible premises to every possible conclusion.*

Proof. See [11]. □

Remark 2.1. *From a conceptual viewpoint here is more fruitful to see why substitution is no longer acceptable into the system throughout an example. For instance let's consider* $A = p \wedge q$*. In order to add that we have to decompose it:*

$$\dfrac{\vdash_B p \qquad \vdash_B q}{\vdash_B p \wedge q}\ (\wedge)$$

Thus let's add the two clauses p *and* q *to the set of beliefs. In classical logic it is possible to substitute* $p \wedge q$ *with a different clause, for example* r *and obtain* $\vdash_B r$*, but* r *is not one of the clauses added to the system and this means that* $[\![r]\!]_B = 0$ *while* $[\![p]\!]_B = [\![q]\!]_B = 1$ *. It's even easier to understand it if the meaning of belief is analyzed. In everyday reasoning it is not possible to substitute beliefs with other beliefs at will and this is why substitution is not acceptable in the system.*

Now it's possible to go further into the formalization of the system: let's define the top sequent made from beliefs added to the system.

Definition 2.3. $top^b(\pi)$: denotes the multiset of all and only π's top-sequents introduced by a belief.

The new way to calculate the value of a sequent will be:

$$[\![A]\!]_B = \frac{top^b(\pi) + top^1(\pi)}{top^b(\pi) + top^1(\pi) + top^0(\pi)}$$

Example 2.1. Let's see an example taking the same sequent seen in example 1.1, but adding now the belief $\vdash_B (p \vee q) \wedge \overline{u}$. The first thing to do, in order to add the belief, is to decompose it and add that to the belief set.

$$\dfrac{\dfrac{\dfrac{}{\vdash_B p, q}\ (b_1)}{\vdash_B p \vee q}\ (\vee) \qquad \dfrac{}{\vdash_B \overline{u}}\ (b_2)}{\vdash_B (p \vee q) \wedge \overline{u}}\ (\wedge)$$

From this it is possible to add the two beliefs $b_1 = p, q$ and $b_2 = \overline{u}$ to the belief set.

Let's take the same decomposition seen in example 1.1: it is possible to see where a belief is added to the system, namely b_1, because $\overline{u}$ doesn't appear in the sequent.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_B p, q}(b_1)}{\vdash_B p \lor q}(\lor) \quad \dfrac{\dfrac{}{\vdash_B p, \overline{p}}(ax.)}{\vdash_B p \lor \overline{p}}(\lor)}{\vdash_B (p \lor q) \land (p \lor \overline{p})}(\land) \quad \dfrac{\dfrac{}{\vdash_B \overline{r}}(\overline{ax.}) \quad \dfrac{}{\vdash_B \overline{t}}(\overline{ax.})}{\vdash_B \overline{r} \land \overline{t}}(\land)}{\vdash_B (p \lor q) \land (p \lor \overline{p}) \land (\overline{r} \land \overline{t})}(\land)$$

This proof contains one identity axiom, one belief and two complementary axioms, so $[\![A]\!]_B$ is:

$$[\![A]\!]_B = \frac{top^b(\pi) + top^1(\pi)}{top^b(\pi) + top^1(\pi) + top^0(\pi)} = \frac{1+1}{1+1+2} = \frac{2}{4} = 0.5$$

Like in example 1.2, it is possible to see the same tree with multi valued system, adding a new rule:

$$\dfrac{}{\mid\!\frac{1}{1}{}_B\ B}(\overline{b_i})$$

Where B denote the set of belief, $B = b_1, \ldots, b_n$.

Example 2.2. Now it's possible to see Example 2.1 with the multi valued system.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mid\!\frac{1}{1}{}_B\ p, q}(b_1)}{\mid\!\frac{1}{1}{}_B\ p \lor q}(\lor) \quad \dfrac{\dfrac{}{\mid\!\frac{1}{1}{}_B\ p, \overline{p}}(ax.)}{\mid\!\frac{1}{1}{}_B\ p \lor \overline{p}}(\lor)}{\mid\!\frac{2}{2}{}_B\ (p \lor q) \land (p \lor \overline{p})}(\land) \quad \dfrac{\dfrac{}{\mid\!\frac{0}{1}{}_B\ \overline{r}}(\overline{\overline{ax.}}) \quad \dfrac{}{\mid\!\frac{0}{1}{}_B\ \overline{t}}(\overline{ax.})}{\mid\!\frac{0}{2}{}_B\ (\overline{r} \land \overline{t})}(\land)}{\mid\!\frac{2}{4}{}_B\ (p \lor q) \land (p \lor \overline{p}) \land (\overline{r} \land \overline{t})}(\land)$$

The only difference between Example 1.2 and 2.2 can be seen in the substitution with the value 1 instead of the 0 for the top axiom $\vdash p, q$. In Figure 1 it is possible to see how the value of A changed when was considered in the fractional semantics framework without beliefs and when one belief is added.

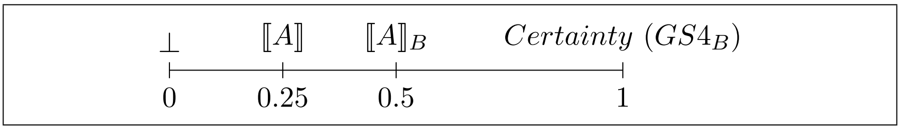

Figure 1: The values of $[\![A]\!]$ and $[\![A]\!]_B$.

It is worth noting that if this sequent was considered in classical logic, it would have different values changing by the value of the atomic formulas, but it would assume value 0 or 1. Something similar happens in Makinson pivotal assumption consequence, also if the belief set is the same that we have defined earlier, because a two valued logic is there considered.

2.1 Strong cut elimination

The last section pointed out that the agent is an ideal one and that they are aware of every deduction between beliefs. This means that the belief set is deductively closed: nothing that was not already in the set can be derived. In order to have a deductively closed belief set it is important that every combination of sentences, when it is possible, must be closed under cut and the new sentences obtained in this way will be added to the belief set.

In order to eliminate cut from $GS4_B$ the method is taken from [13], but it is simplified because of the nature of one-sided sequents. The method is the following:

1. turn each new belief b_i added to the system in a conjunctive form: $\text{cnf}(b_i) = b_1 \wedge \cdots \wedge b_n$ and add to the system each of the atomic formulas;
2. for each disjunctive formula, let's remove the application of $\vee$-rule: $b_1 \vee \cdots \vee b_n \equiv b_1, \ldots, b_n$;

3. let's remove copies of the same sequent;

4. let's remove identity sequents of the form $\vdash \Gamma, p, \overline{p}$;

5. let's close under cut the belief set and add to the system the new formulas obtained in this way.

It is easy to show why the last step is so important. Suppose that an agent has a new belief: $A = (\overline{p} \wedge (\overline{t} \vee q)) \vee (t \wedge (\overline{t} \vee q))$. The first thing to do in order to add that belief is to transform A in a conjunctive form: it is easy to show that it is equivalent to $\vdash (\overline{p} \vee t) \wedge (\overline{t} \vee q) \wedge (t \vee \overline{t} \vee q) \wedge (\overline{t} \vee q)$. Let's decompose it in a set of clauses: $\vdash \overline{p}, t$, $\vdash \overline{t}, q$, $\vdash t, \overline{t}, q$, $\vdash \overline{t}, q$ and remove one of the copies of $\vdash \overline{t}, q$ and the axiom $\vdash t, \overline{t}, q$. By the method presented earlier the agent has to add $(\overline{p} \vee t)$ and $(\overline{t} \vee q)$ to the system, but these beliefs are not cut free. To let them be cut free, it is necessary to close them under the cut.

$$\frac{\vdash \overline{p}, t \qquad \vdash \overline{t}, q}{\vdash \overline{p}, q}\ (cut)$$

From the last point of the method presented earlier, it is needed to add not only $\vdash \overline{p}, t$ and $\vdash \overline{t}, q$, but also $\vdash \overline{p}, q$. Let's see why: $[\![(\overline{p} \vee t) \wedge (\overline{t} \vee q)]\!]$ has value 1 if $B = \{(\overline{p}, t); (\overline{t}, q)\}$

$$\frac{\dfrac{\models^{1}_{1\,B} \overline{p}, t}{\models^{1}_{1\,B} \overline{p} \vee t}\ (\vee) \qquad \dfrac{\models^{1}_{1\,B} \overline{t}, q}{\models^{1}_{1\,B} \overline{t} \vee q}\ (\vee)}{\models^{2}_{2\,B} (\overline{p} \vee t) \wedge (\overline{t} \vee q)}\ (\wedge)$$

But what about $(\overline{p} \vee t) \wedge (\overline{t} \vee q) \wedge (\overline{p} \vee q)$?

Remark 2.2. $(\overline{p} \vee t) \wedge (\overline{t} \vee q) \wedge (\overline{p} \vee q)$ *is classically equivalent to* $(\overline{p} \vee t) \wedge (\overline{t} \vee q)$, *this means that they must have the same value because of stability, i.e.* $[\![(\overline{p} \vee t) \wedge (\overline{t} \vee q)]\!] = [\![(\overline{p} \vee t) \wedge (\overline{t} \vee q) \wedge (\overline{t} \vee q)]\!]$.

What happens if $B = \{(\overline{p}, t); (\overline{t}, q)\}$ is considered, instead of $B = \{(\overline{p}, t); (\overline{t}, q), (\overline{p}, q)\}$, obtained by adding also the belief closed under cut? Let's consider the decomposition of $(\overline{p} \vee t) \wedge (\overline{t} \vee q) \wedge (\overline{p} \vee q)$ within the set $B = \{(\overline{p}, t); (\overline{t}, q)\}$.

$$\dfrac{\dfrac{\dfrac{\vdash^{1}_{1\,B} \overline{p}, t}{\vdash^{1}_{1\,B} \overline{p} \vee t}\,(\vee) \qquad \dfrac{\vdash^{1}_{1\,B} \overline{t}, q}{\vdash^{1}_{1\,B} \overline{t} \vee q}\,(\vee)}{\vdash^{2}_{2\,B} (\overline{p} \vee t) \wedge (\overline{t} \vee q)}\,(\wedge) \qquad \dfrac{\vdash^{0}_{1\,B} \overline{p}, q}{\vdash^{0}_{1\,B} \overline{p} \vee q}\,(\vee)}{\vdash^{2}_{3\,B} (\overline{p} \vee t) \wedge (\overline{t} \vee q) \wedge (\overline{p} \vee q)}\,(\wedge)$$

This way a different value for the sequent is obtained and it must have the same value of $\vdash (\overline{p} \vee t) \wedge (\overline{t} \vee q)$.

This means that it is important to pay attention to frame not only the axioms obtained by the decomposition of the sequent, but also every formula closed under cut. In fact if the set $B = \{(\overline{p}, t); (\overline{t}, q), (\overline{p}, q)\}$ is considered, the original value is restored.

$$\dfrac{\dfrac{\dfrac{\vdash^{1}_{1\,B} \overline{p}, t}{\vdash^{1}_{1\,B} \overline{p} \vee t}\,(\vee) \qquad \dfrac{\vdash^{1}_{1\,B} \overline{t}, q}{\vdash^{1}_{1\,B} \overline{t} \vee q}\,(\vee)}{\vdash^{2}_{2\,B} (\overline{p} \vee t) \wedge (\overline{t} \vee q)}\,(\wedge) \qquad \dfrac{\vdash^{1}_{1\,B} \overline{p}, q}{\vdash^{1}_{1\,B} \overline{p} \vee q}\,(\vee)}{\vdash^{3}_{3\,B} (\overline{p} \vee t) \wedge (\overline{t} \vee q) \wedge (\overline{p} \vee q)}\,(\wedge)$$

Thus the sequents have the same value:

$$[\![(\overline{p} \vee t) \wedge (\overline{t} \vee q)]\!]_B = [\![(\overline{p} \vee t) \wedge (\overline{t} \vee q) \wedge (\overline{p} \vee q)]\!]_B = 1$$

As it was showed, the cut is really important for a complete set of beliefs, but it is also necessary to see how the cut can be eliminated from the calculus.

Elimination of cut The elimination of cut in presence of proper axioms was firstly proposed by Girard [4], as noted by Avron [1], upgrading the Gentzen's standard cut elimination algorithm. The procedure here proposed, i.e., the decomposition of the formula, the add to the system and the cut of the formula to obtain all the derivations, owes a lot to the one presented in [13]. In the article, in fact, is proved that, for any cluster of extra-logical assumptions, there exists exactly one axiomatic extension of classical propositional logic that admits cut elimination. First of all it is possible to see that weakening it is admissible in $GS4_B$.

Theorem 2.2 (Weakening admissibility in $GS4_B$)**.** *For two atomic formulas* $\vdash_B \Gamma$ *and* $\vdash_B \Delta$*,* $[\![\bigvee \Gamma \vee \bigvee \Delta]\!]_B \geq [\![\bigvee \Gamma]\!]_B$*.*

Proof. To prove this is sufficient to consider a transformation of $\vdash_B$. In fact if $B = b_1, \ldots b_n$, then $\vdash_B \Gamma$ is equal to $\vdash \Gamma, \overline{b_1}, \ldots \overline{b_n}$ with the classical closure as pointed out in [11][1]. □

It is possible to generalize this result for any context:

Theorem 2.3. *For any context* Γ *and a formula* A*, such that* A *is not contradictory with the set* B*,* $[\![\bigvee \Gamma \vee A]\!]_B \geq [\![\Gamma]\!]_B$*.*

Proof. By induction on the complexity of the formula A. □

The proof shows that this system is totally monotonic, also if it could seem counterintuitive for a set of belief., because maybe if an agent has a belief p it is strange to believe also $p \wedge q$, but this is due to the fact that this work is based on a classical framework and so the fractional value of $[\![p]\!]_B$ assumes the same of $[\![p \wedge q]\!]_B$.

Theorem 2.4 (Strong cut elimination of $GS4_B$)**.** *The cut rule is redundant when added to* $GS4_B$*.*

Proof. Similar to the one proposed in [13]. □

[1] In the text the two sided version of this transformation was used, so $\vdash_B \Gamma$ becomes $b_1, \ldots, b_n \vdash \Gamma$, but here because of the choice to use $GS4$ as main system, it is used the one-sided classically equivalent version $\vdash \Gamma, \overline{b_1}, \ldots, \overline{b_n}$.

The set of beliefs can be "completed" through cut or without that. This means that $GS4_B$ is a cut-free system, because it is an axiomatic extension of classical logic. By the way, the use of cut can alter the fractional semantics value as shown in [12]. Thanks to theorem 2.4 the algorithm presented in section 2.1 can be transformed in an algorithm without the presence of cut. As a corollary of the strong cut elimination it can be obtained:

Theorem 2.5 (Uniqueness of axiomatization in $GS4_B$)**.** *For any cluster of axioms in the set of beliefs B the axiomatization is unique.*

Proof. See [13]. □

3 Conclusions

The present research may be considered as an attempt to create a bridge between logic and the real world, nonetheless retaining consistency and decidability.

The main difference between classical approach and the fractional semantics one is either philosophical and technical. By a technical point of view is possible to distinguish between different levels of contradiction. Fractional semantics is able to distinguish between what's inside truthfulness and falsity, without losing the rigorous approach to proof theory, proving the cut elimination and the others fundamental properties.

The system $GS4_B$ is able to consider beliefs and logical axioms together. The fractional system, in fact, does not have the symmetry between tautologies and contradictions and it is helpful to talk about something that is uncertain such as beliefs. We have shown that fractional semantics could be useful to see how to solve a simple, but tricky problem such as the Lottery Paradox.

It's interesting to see how fractional semantics behaves differently compared to classical interpretation and probability. It was important to point out the difference between probability and fractional semantics, also because it will not be strange that fractional semantics and probability will be able to work together to better define logical derivations involving beliefs.

Further researches This kind of implementation could be expanded, for example considering restrictions of the set of beliefs, i.e., applying fractional semantics to non monotonic logics or non classical logics in general.

In the Paradox of lottery it was interesting to notice that these different results, interpreted together, can be richer in content than probability itself, letting know, through a value, the number of true and false conjuncts propositions. Despite this, a way to let probability, fractional semantics and maybe belief revision work together must be implemented yet.

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