A Formal Proof-Theoretic Approach to Music Analysis using Labelled Lambek Calculus

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Abstract: Music and language are thought to share a common origin, leading to numerous research endeavors that have sought to analyze music through a compositional approach, grounded in linguistic grammar rules. In this study, we extend and refine this method, demonstrating that the analysis can be systematically represented using rigorous proof theory, akin to how formal logic elucidates natural language semantics. Our approach involves the use of sequent calculus to construct a proof, notably extending Lambek calculus for categorial grammar to a labeled version. The introduction of labels enables each term within a sequent to be interpreted as a chord with a specified key (tonality).

Keywords: Labelled Lmbek Calculus, Chord sequence analysis

1. Introduction

As Charles Darwin noted, the origins of human language and music are believed to be the same, used for wooing between primordial males and females[17]. Since natural languages are context-free languages (CFL) in the Chomsky hierarchy, their syntax forms a tree structure. If language and music share a common origin, music should also have a hierarchical tree structure.

Generative Theory of Tonal Music (GTTM) [10], Rohrmeier's Generative Syntax Model (GSM) [14], and others have demonstrated tree-structured analyses of music. Among these, Steedman et al. [6] used combinatorial category grammar (CCG).

In this paper, we extend music analysis beyond syntax to formal semantics. Similar to language semantics in logical formalism, we interpret music through logic. We analyze a piece of music by interpreting a sequence of chord names as pairs of key and degree, composing these interpretations structurally, and ultimately reducing them to a cadence.

In this work, we use structural proof theory, specifically Troelstra's Basic Proof Theory [16], and explain the construction of a sequence of chords with sequent calculus [1]. Structural proof theory dates back to the early 1930s when Gerhard Gentzen introduced natural deduction and sequent calculus in his thesis [7]. Gentzen's Hauptsatz theorem states that any proof in classical logic can be transformed into a cut-free form, demonstrating the consistency of the system.

We employ labelled sequent calculus, where the label represents a possible world, indicating the region of tonality or mode. While modal logic has been used for future expectations in music progression [15], we regard modality as a key or region of the diatonic set.

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This paper is organised as follows: in Section 2, we set up our formalism. In the following Section 3, we'll show an expansion of the analysis to the Bartòk's axis system. In Section 4, we show that the analysis process can also be viewed as its dual. Finally, we summarise our contribution and limitations and discuss future issues.

2. Labelled calculus for music analysis

2.1 Chord Notation and Lexicon

First, we strictly distinguish the three kinds of chord notations.

Berklee Chord names Em7, A7, Cm7, etc, each of which is a set of diatonic notes, and are shown in upright fonts.

Key : Degree A pair of a key (tonality) and a degree in roman numeric is shown by being connected by colon (:), e.g., C:I, G:iii, a:i, etc. The lowercase in key is a minor, and the lowercase numeric is a minor third, and they are shown in italic fonts.

Chord functions **T** (tonic), and **D** (dominant), **S** (subdominant), shown in boldface fonts.

We provide a *Lexicon*, where a Berklee chord is looked up and it can be *interpreted* in multiple ways, as follows.

F	$\Rightarrow$	$C:\mathrm{IV},\ F:\mathrm{I},\ B\flat:\mathrm{V},\cdots$
G	$\Rightarrow$	$G:\mathrm{I},\ D:\mathrm{IV},\ a:\mathrm{VII},\cdots$
B♭	$\Rightarrow$	$F:\mathrm{IV},\cdots$
C7	$\Rightarrow$	$F:\mathrm{V7},\cdots$
⋮		⋮

2.2 Labelled Lambek Calculus (LLC)

Lambek Calculus (LC) is a sequent calculus for Categorial Grammar (CG). Since CG can bind an adjacent word or a category either from the left-hand or the right-hand side, we need to provide rules in two different directions: →' and ←'.*1 In Kripke semantics of modal logic, $\Box x$ means the necessity; that is, in all the accessible possible worlds, x must hold, and $\Diamond x$ means that there exists some possible worlds in which x holds. In Table 1, we show a set of general rules for labelled Lambek calculus (LLC),*2 apart from the music construction.

2.3 LLC rules for cadences

Hereafter, we use $\alpha, \beta, \cdots$ for keys, and $x, y, \cdots$ for degrees. Now, we employ LLC for cadence rules. We regard $\alpha R \beta$ is a key modulation from α to β, that is, key β is accessible from α if they are related. For example, if R represents a shift to the parallel key, chord $\alpha : x$ becomes $\beta : y$ in the parallel key. R works in a different way, according to shifts to parallel, relative, dominant, and sub-dominant keys.

We include the following chord re-interpretation as an initial sequent:

$$\alpha : x, \alpha R \beta \Rightarrow \beta : R(x),$$

together with other initial sequents for chord functions:

$$\alpha : \mathrm{I} \vdash \alpha : \mathbf{T},\ \alpha : \mathrm{V} \vdash \alpha : \mathbf{D},\ \alpha : \mathrm{IV} \vdash \alpha : \mathbf{S}, \text{and so on.}$$

We introduce Cadence Rules below, instantiating the labelled Lambek rules in Table 1.

$$\frac{\Gamma \vdash \alpha : \mathbf{S} \qquad \Delta, \alpha : \mathrm{V} \vdash \alpha : \mathbf{D}}{\Delta, \Gamma, \alpha : \mathbf{S} \to \mathrm{V} \vdash \alpha : \mathbf{D}}\ (\to_L)$$

$$\frac{\Gamma \vdash \alpha : \mathbf{D} \qquad \Delta, \alpha : \mathrm{I} \vdash \alpha : \mathbf{T}}{\Delta, \Gamma, \alpha : \mathbf{D} \to \mathrm{I} \vdash \alpha : \mathbf{T}}\ (\to_L)$$

$$\frac{\alpha : \mathrm{I} \vdash \alpha : \mathbf{T} \qquad \Gamma, \alpha : \mathrm{I} \vdash \alpha : \mathbf{T}}{\Gamma, \alpha : \mathbf{T} \leftarrow \mathrm{I}, \alpha : \mathrm{I} \vdash \alpha : \mathbf{T}}\ (\leftarrow_L)$$

From the given initial sequents and instantiated rules, the right-hand side of '⊢' is always a chord function; it means that a sequence of chords is *interpreted* to bear a function.

For an easy example, given an input chord sequence: G–C–D–G, we can obtain the following construction, consulting the lexicon.

$$\dfrac{\dfrac{\dfrac{\mathrm{C} \Rightarrow G{:}\mathrm{IV}}{G : \mathrm{IV} \vdash G : \mathbf{S}} \qquad \dfrac{\mathrm{D} \Rightarrow G{:}\mathrm{V}}{G : \mathrm{V} \vdash G : \mathbf{D}}}{G : \mathbf{S} \to \mathrm{V}, G : \mathrm{V} \vdash G : \mathbf{D}} \qquad \dfrac{\mathrm{G} \Rightarrow G{:}\mathrm{I}}{G : \mathrm{I} \vdash G : \mathbf{T}}}{G : \mathbf{D} \to \mathrm{I}, G : \mathrm{I} \vdash G : \mathbf{T}\ (\dagger)}\ (\to_L)$$

$$\dfrac{\dfrac{\mathrm{G} \Rightarrow G{:}\mathrm{I}}{G : \mathrm{I} \vdash G : \mathbf{T}} \qquad (\dagger)}{G : \mathbf{T} \leftarrow \mathrm{I}, G : \mathrm{I} \vdash G : \mathbf{T}}\ (\to_L)$$

However when the chord C or D occupy longer time duration, we should regard they are local modulations as C:I or D:I. In which case, the latter part of a valid proof tree is constructed, as follows.

$$\frac{\mathrm{C} \Rightarrow C{:}\mathrm{I} \qquad C : \mathrm{I},\ {}_{C}R_{G} \Rightarrow G : \mathrm{IV}}{G : \mathrm{IV} \vdash G : \mathbf{S}\ (\ddagger 1)}$$

$$\frac{\mathrm{D} \Rightarrow D{:}\mathrm{I} \qquad D : \mathrm{I},\ {}_{D}R_{G} \Rightarrow G : \mathrm{V}}{G : \mathrm{V} \vdash G : \mathbf{D}\ (\ddagger 2)}$$

*1 In the original LC, '/' and '\' are used for two directions, but '/' is confusing in the context of chord representation ('/' is used for *doppel*-dominant or to denote a bass in a chord). Therefore, we employ arrows instead.

*2 The original LC includes a construction by dot (·), however, we omit them here since they are not mentioned in this paper.

$$\dfrac{\dfrac{(\ddagger 1) \qquad (\ddagger 2)}{G : \mathbf{S} \to \mathrm{V}, G : \mathrm{G} \vdash G : \mathbf{D}} \qquad \dfrac{\mathrm{G} \Rightarrow G{:}\mathrm{I}}{G : \mathrm{I} \vdash G : \mathbf{T}}}{G : \mathbf{D} \to \mathrm{I}, G : \mathrm{I} \vdash G : \mathbf{T}}$$

An useful application of this kind of method can be seen in the analysis of "Stella by Starlight" [2], we report here the central part as an interesting example, see Table 2.3.

3. Expanding the analysis

It is possible to change the rules of the system in order to investigate a different way of writing music. For example Bartòk used to write in a way called Axis System [9], i.e., a system where Tonic, Subdominant and Dominant are made thanks to the axis on the tritone as shown in Figure 1.

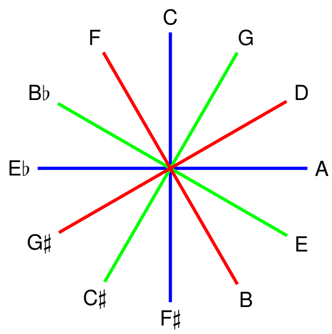


Fig. 1 The axis system is divided in tonic, in blue, subdominant, in red and dominant, in green.

Expanding the system means that we can associate $\alpha, \beta, \ldots$ to $\mathbf{T}$ not only for the real tonic, but also for the chords in the axis. The first movement of *Music for Strings, Percussion, and Celesta*, for example, begins in the key of A minor, denoted as $a : \mathbf{T}$. At the golden section of the piece (measure 56), the tonality shifts to $E\flat$, which is denoted as $E\flat : \mathbf{T}$. In the second movement, the tonalities of C and F♯ are also presented, and all of these keys are denoted as Tonic. These considerations, purely based on musical praxis, can lead us to adopt rules that were previously considered false, such as:

$$\dfrac{\dfrac{\dfrac{\mathrm{Bm} \Rightarrow \mathrm{C}{:}\ VIIm}{\mathrm{C}{:}\ VII \vdash \mathrm{C}{:}\ \mathbf{S}} \qquad \dfrac{\mathrm{C}\sharp \Rightarrow \mathrm{C}{:}\ I\sharp}{\mathrm{C}\sharp{:}\ V \vdash \mathrm{C}\sharp{:}\ \mathbf{D}}}{\mathrm{C}{:}\ \mathbf{S} \to V, \mathrm{C}{:}\ V \vdash \mathrm{C}{:}\ \mathbf{D}}\ (\to_L) \qquad \dfrac{\mathrm{C_{MA}}^7 \Rightarrow \mathrm{C}{:}I}{\mathrm{C{:}I} \vdash \mathrm{C}{:}\mathbf{T}}}{{}^{*(4)}\mathrm{C}{:}\ \mathbf{D} \to I, \mathrm{C}{:}\ I \vdash \mathrm{C}{:}\ \mathbf{T}}\ (\to_L)$$

Where C♯ is considered a dominant since it is on the axis of the tonic (i.e., the green one in Figure 1). We believe that this example better demonstrates the malleability of the system we have created.

4. Composition Algorithm for Labelled Sequents

The idea of Composition Algorithm for LLS is similar to the one presented in [1], but it will be a little more intricate due to the non-linearity of harmony. For example, the first 11 chords of "Stella by Starlight" can be notated in this manner:

In the upper part, we have connected all the Tonic chords to their respective counterparts. In the lower part, we have established connections for the other functions, including Subdominant and Dominant chords. This comprehensive representation

$$\frac{\Gamma \vdash \alpha : x \qquad \Delta, \alpha : y, \Sigma \vdash \beta : z}{\Delta, \alpha : y \leftarrow x, \Gamma, \Sigma \vdash \beta : z}\ (\leftarrow_L) \qquad \frac{\Gamma, \alpha : x \vdash \alpha : y}{\Gamma \vdash \alpha : y \leftarrow x}\ (\leftarrow_R) \qquad \frac{\Gamma \vdash \alpha : \Box x}{\alpha R \beta,\ \Gamma \vdash \beta : x}\ (\Box)$$

$$\frac{\Gamma \vdash \alpha : x \qquad \Delta, \alpha : y, \Sigma \vdash \beta : z}{\Delta, \Gamma, \alpha : x \rightarrow y, \Sigma \vdash \beta : z}\ (\rightarrow_L) \qquad \frac{\alpha : x, \Gamma \vdash \alpha : y}{\Gamma \vdash \alpha : x \rightarrow y}\ (\rightarrow_R) \qquad \frac{\alpha R \beta,\ \Gamma \vdash \beta : x}{\Gamma \vdash \alpha : \Diamond x}\ (\Diamond)$$

$$\frac{\Gamma \vdash \alpha : x \qquad \alpha : x \vdash \beta : y}{\Gamma \vdash \beta : y}\ (\mathrm{Cut})$$

Table 1 Labelled Lambek Calculus, where the right-hand side of '⊢' is intuitionistically restricted to only one term. (□) tells that if $\alpha R \beta$ deduces $\beta : x$ then $\alpha : \Box x$, and (◇) vice versa. Note that $\alpha R \beta$ appears order-free in sequents.

14-16

$$\dfrac{\dfrac{\dfrac{\mathrm{Em7}^{(\flat 5)} \Rightarrow \mathrm{d}{:}\ II \qquad \mathrm{d}{:}\ II,\ {}_d\mathrm{R}_D \Rightarrow \mathrm{D{:}II}}{\mathrm{D}{:}II \vdash \mathrm{D}{:}\ \mathbf{S}} \qquad \dfrac{A7 \Rightarrow \mathrm{D}{:}\ V}{\mathrm{D}{:}\ V \vdash \mathrm{D}{:}\ \mathbf{D}}}{\mathrm{D}{:}\ \mathbf{S} \rightarrow V, \mathrm{D}{:}\ V \vdash \mathrm{D}{:}\ \mathbf{D}}\ (\rightarrow_L) \qquad \dfrac{\mathrm{D7} \Rightarrow \mathrm{D}{:}\ I}{\mathrm{D}{:}\ I \vdash \mathrm{D}{:}\ \mathbf{T}}}{{}^{*(4)}\mathrm{D}{:}\ \mathbf{D} \rightarrow I, \mathrm{D}{:}\ I \vdash \mathrm{D}{:}\ \mathbf{T}}\ (\rightarrow_L)$$

15-17

$$\dfrac{\dfrac{\dfrac{\mathrm{Am7}^{(\flat 5)} \Rightarrow \mathrm{g}{:}\ II \qquad \mathrm{g}{:}\ II,\ {}_g\mathrm{R}_{G^{(\sharp 5)}} \Rightarrow \mathrm{G}^{(\sharp 5)} : II}{\mathrm{G}^{(\sharp 5)} : II \vdash \mathrm{G}^{(\sharp 5)} : \mathbf{S}} \qquad \dfrac{D7 \Rightarrow \mathrm{G}^{(\sharp 5)} : V}{\mathrm{G}^{(\sharp 5)} : V \vdash \mathrm{G}^{(\sharp 5)} : \mathbf{D}}}{\mathrm{G}^{(\sharp 5)} : S \rightarrow V, \mathrm{G}^{(\sharp 5)} : V \vdash \mathrm{G}{:}\ \mathbf{D}}\ (\rightarrow_L) \qquad \dfrac{\mathrm{G}^{(\sharp 5)} \Rightarrow \mathrm{G}^{(\sharp 5)} : I}{\mathrm{G}^{(\sharp 5)} : I \vdash \mathrm{G}^{(\sharp 5)} : \mathbf{T}}}{\mathrm{G}^{(\sharp 5)} : \mathbf{D} \rightarrow I, \mathrm{G}^{(\sharp 5)} : I \vdash \mathrm{G}^{(\sharp 5)} : \mathbf{T}}\ (\rightarrow_L)$$

17-20

$$\dfrac{\dfrac{\mathrm{G}^{(\sharp 5)} \Rightarrow \mathrm{G}^{(\sharp 5)} : I \qquad \mathrm{G}^{(\sharp 5)} : I,\ {}_{G^{(\sharp 5)}}\mathrm{R}_c \Rightarrow \mathrm{c}{:}\ V}{\mathrm{c}{:}\ V \vdash \mathrm{c}{:}\ \mathbf{D}} \qquad \dfrac{\mathrm{Cm7} \Rightarrow \mathrm{c}{:}I}{\mathrm{c{:}I} \vdash \mathrm{c}{:}\mathbf{T}}}{\mathrm{c}{:}\ \mathbf{D} \rightarrow I, \mathrm{c}{:}\ I \vdash \mathrm{c}{:}\ \mathbf{T}}\ (\rightarrow_L)$$

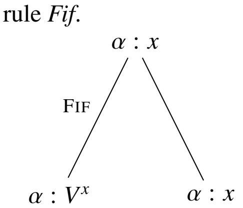
Table 2 Example from "Stella By Starlight", the measures are indicated on the left of the trees.

allows us to visualize the harmonic relationships more thoroughly and understand the interplay between different chord functions. It can alternatively be expressed in the form of chord functions:

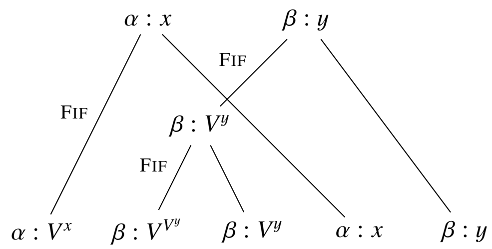
There are two main reasons to use this method: it provides a clear map of the harmonic sequence, though less expressive than sequents, and it aids in composition by creating the dual of the rules introduced for LLC.

4.1 Rules

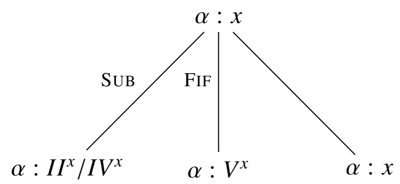
Music's flexible nature allows easy modification of these rules. For instance, we can start with a simple rule: from any chord, adding a dominant is always possible to maintain a tonal structure. Let's call this rule *Fif*.



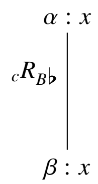
where V^x indicates the dominant of the degree x. Differently from the one presented in [1] here the dominants can be chained:



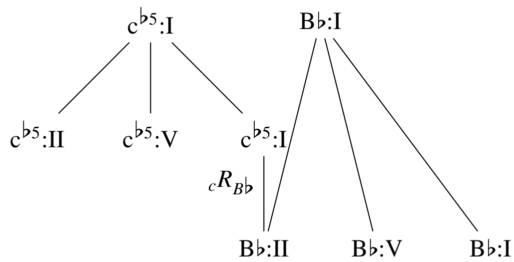
Here, we encounter three instances of the rule *Fif*, and they are intertwined. Obviously we can chain that to every $\alpha : S$ ($\alpha : II$ or $\alpha : IV$), let's call it *Sub*.



Furthermore, cadences can introduce variations in the function of specific chords. For instance, a C7 chord can serve as the dominant of $\mathrm{F}\textsc{ma}^7$, yet it may also function as the first degree in a cadence that resolves to G7. An illustration of this concept can be found in the final part of Stella By Starlight (see Figure 2, that can be rewritten easily as in Figure 3), where the harmonic progression takes an unexpected turn. In general we can write that:



This implies that when there is an accessibility relation ${}_\alpha R_\beta$, we have the flexibility to change the interpretation of the chord. For instance, the final part of "Stella By Starlight," it can be seen as:



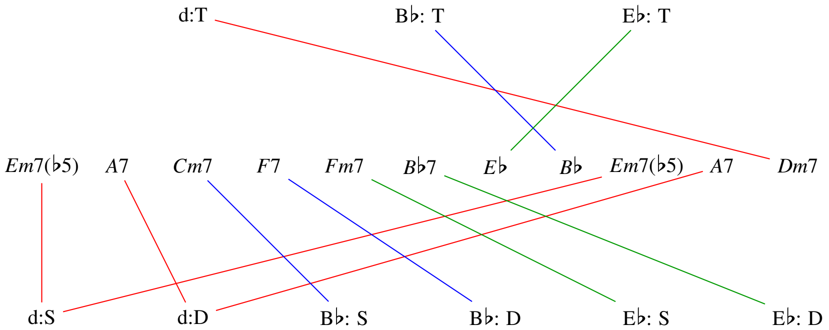


Fig. 2 Using the Composition Algorithm, it is also possible to show the analysis. In this case, we can find the first measures of "Stella by Starlight". In the upper part, the tonics are displayed; in the central part, the chords; and in the lower part, the functions. The colored lines connect the tonalities, the chords, and the functions.

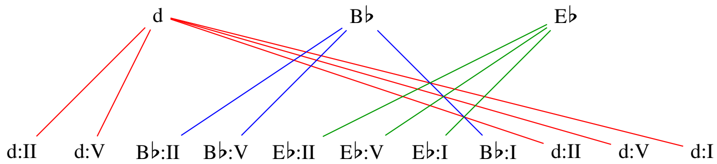
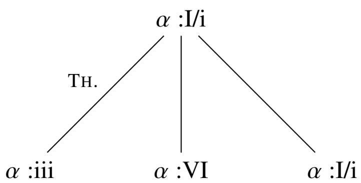


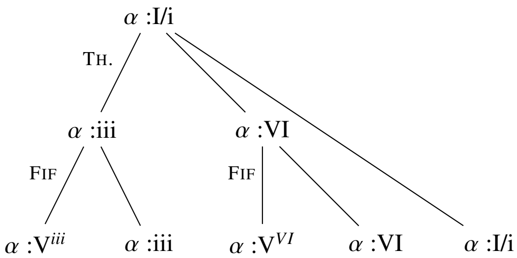
Fig. 3 This figure represents the analysis of "Stella By Starlight" again, but in a more readable way. In the upper part, we have the tonalities, and in the lower part, the chords are written as functions.

4.2 New rules

We didn't want to be complete in the list of the rules, in fact the flexibility of this system proves to be useful when incorporating new rules. For instance, if we wish to introduce a rule allowing the tonic (and only the tonic) to split into its third minor and sixth major, it can be easily formalized in the following manner, calling it *Th.*:



Obviously these rules can be combined into new harmonic structures:



The significant aspect of this work is that such structures are inherently rational; that is, they adhere to well-established rules or rules determined by the composer. This implies that, on one hand, the structure can be very strict, while on the other hand, it can also be highly malleable.

5. Conclusions

Based on the traditional view that language and music share a common origin, we have treated a sequence of chord names as a surface sentence and developed a structural proof system with sequent rules to determine grammaticality. Previous works have shown syntactic tree structures for chord sequences, but our approach extends this by providing semantics, interpreting each chord, and deriving chord functions. Our new methodology employs labelled sequent calculus. Gentzen's sequent calculus, known for its hierarchical composition adequacy, is the basis for Lambek calculus (LC), a sequent version of categorial grammar (CG).

In a previous work [2] we have analyzed Stella By Starlight using this method, demonstrating its effectiveness. In this paper we have laid the foundations for a comprehensive analysis of music adding the Bartòk's axis sytem to the Lambek Calculus Analysis. The system shows that harmony is not a linear process, requiring a non-linear analysis solvable through labelled sequent calculus. We also explored the system's application in composition, utilizing the dual of sequents.

Music styles vary by era and genre, leading to many variations in authentic theory. Our system is adaptable, allowing different sets of sequent rules, though their adequacy must be objectively verified. Since some rules are rarely used, their application probability should also be considered. Currently, we have used labels only for key-shift but should incorporate their original meanings (necessity and possibility). Future work includes automating this analysis system and potentially creating an automatic composer system based on our composition algorithm.

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