Wittgenstein, probability and supraclassical logics

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Abstract

In his *Tractatus*, Wittgenstein proposed a method for calculating probability using truth tables, which served as inspiration for Carnap and Ramsey's work on probability. Despite this, Wittgenstein's idea was not widely considered in the literature. Some scholars have interpreted Wittgenstein's idea as a generalization of the indifference principle, while others view it as an attempt to analyze the relationship between beliefs and logic. Wittgenstein's method involves comparing two propositions, where the first is considered only in true instances, while the other is analyzed only when the first is true. This approach is not dissimilar from Makinson's supraclassical logic, despite the use of different methods.

The aim of this work is to shed light on Wittgenstein's method and demonstrate the relationship between Wittgenstein's probability and Makinson's supraclassical logic, illustrating that Wittgenstein created one of the first logics that was able to consider beliefs in the calculus.

Wittgenstein's method involves comparing two propositions, one of which is only considered in true instances, while the other is analyzed only when the first is true. The probabilities of these propositions are then compared to determine their relationship. This approach is similar to Makinson's supraclassical logic, which considers statements together with set of beliefs. We will also address an intriguing link with the Lottery paradox.

In conclusion, Wittgenstein's method for calculating probability using truth tables, while not widely considered in the literature, was an important contribution to the development of probability theory. Wittgenstein's work demonstrates the importance of considering beliefs when reasoning about probability, and his contributions continue to be relevant to contemporary discussions in the field.

1 Introduction

The aim of this work is to provide a comprehensive account of Wittgenstein's notion of probability, as originally introduced in the Tractatus and linking that to supraclassical logics. While some have interpreted Wittgenstein's understanding of probability as merely a generalization of the indifference principle, we argue that it encompasses a far more nuanced perspective. Although Wittgenstein's initial interpretation of probability may appear simplistic, a more thorough analysis reveals its depth and significance. Briefly, it is one of the first attempts to consider logic within beliefs.

According to Wittgenstein, probability is defined by the relationship between the "belief's truth-possibilities" (*Wahrheitsmöglichkeiten*) [1, 2, 9, 13] and the truth possibilities of the proposition under consideration. Throughout his Tractatus [19], Wittgenstein posits that probability is *a priori* and maintains this viewpoint in his later writings, wherein he firmly rejects *frequentism* as the correct interpretation of probability:

> Let's assume that someone playing dice every day were to throw, say, nothing but ones for a whole week, and that he does this with dice that turn out to be good when subjected to all other methods of testing, and that also produce the normal results when someone else throws them. Does he now have reason to assume a natural law here, according to which he always has to throw ones? Does he have reason to believe that things will continue in this way – or (rather) to assume that this regularity won't last much longer? So does he have reason to quit the game since it has turned out that he can throw only ones; or to continue playing, because now it is just all the more likely that on the next try he'll throw a higher number? – In actual fact he'll refuse to acknowledge the regularity as a law of nature; at least it will have to last for a long time before he'll consider this view of regularity. But why? – I think it's because so much of his previous experience in life refutes such a law, experience that has to be, so to speak – vanquished before we accept a totally new way of looking at things. [18, 104e]

One of the objectives of this paper is to prove that Wittgenstein's probability is a supraclassical logic, i.e., a logic that is able to derive more than classical logic usually permits [12], that is able to consider beliefs as axioms. In this sense it is mandatory to lose substitution due to the Post completeness [16]. The idea lies on the fact that for Wittgenstein probability is a sort of extension of classical logic:

> [5.156] It is in this way that probability is a generalization.
> It involves a general description of a propositional form.
> We use probability only in default of certainty—if our knowledge of a fact is not indeed complete, but we do know something about its form.
> (A proposition may well be an incomplete picture of a certain situation, but it is always a complete picture of something.)
> A probability proposition is a sort of excerpt from other propositions.

In Wittgenstein's concept of probability, truthfulness is not solely determined by logic but also by *knowledge*, namely, beliefs. As a result, propositions that do not hold as true in classical logic due to not being tautologies can still be assigned values greater than 0 in probability framework. The objective of this research is to establish a connection between Wittgenstein's idea of probability as a supraclassical logic and the development of his views on probability.

To accomplish this, we draw upon Makinson's foundational work, "Bridges between classical and non-monotonic logic" [12], with a specific focus on the section dedicated to probability and beliefs. This examination seeks to illustrate that Wittgenstein's probability, albeit unconventional, aligns with supraclassical logics and with Kolmogorov's axioms, signifying its compatibility with fundamental principles of probability theory. The peculiar aspect of Wittgenstein's probability should be noted, as it extends beyond the confines of classical logic by incorporating knowledge-based evaluations of truthfulness for propositions.

Moreover, we provide a link between the well known Lottery Paradox and the Wittgenstein's idea of probability, showing that in this framework it is easily solvable, but also interesting from the philosophical point of view. By delving into these connections, we aim to shed light on the unique characteristics and implications of Wittgenstein's probabilistic approach.

2 Probability and possibility in Wittgenstein's *Tractatus*

Wittgenstein's probability can be seen as an endeavor to establish a connection between beliefs and propositions. In "theories of probabilities" [3], De Finetti distinguishes between possibilities, which are objective, and probabilities, which are subjective. Wittgenstein's approach lies somewhere in between these two concepts. On one hand, Wittgenstein analyzes each possibility of falsity and truthfulness, akin to possibilities in De Finetti's framework. However, on the other hand, the agent's ability to choose the initial set of propositions introduces a subjective element, linked to the agent's personal knowledge of a given argument.

Despite its significance, probability in Wittgenstein's work is often considered marginal, with little written on his specific views regarding probability. Notably, Wittgenstein's primary reflections on the nature of probability can be found in the Tractatus, starting from proposition 5.1.

> [5.1] Truth-functions can be arranged in series.
> That is the foundation of the theory of probability

Surprisingly, Von Wright, one of the first and most important authors that worked on Wittgenstein and in particular on the topic of probability in his work, does not include this proposition in the list of meaningful propositions about probability in [20], where he states:

> There are three main sources for a study of Wittgenstein's views of probability. The first are propositions 5.15 — 5.156 of the *Tractatus*. The second is Section XXII of the Philosophische Bemerkungen written in 1929 or 1930. The third is a typescript of 18 pages, presumably composed in the academic year 1932-1933 on the basis of manuscripts from the immediately preceding years. [20, p. 259]

In summary, Wittgenstein's probability can be viewed as an intriguing attempt to bridge the gap between beliefs and propositions. While resembling objective possibilities in some aspects, it also exhibits subjective characteristics by allowing agents to determine the initial set of propositions based on their personal knowledge. Despite its relative lack of extensive treatment, Wittgenstein's reflections on probability in the Tractatus offer valuable insights into this complex subject. Preliminar reflections on these can be found in the Notebooks 1914-1916 and in the reflections made in the *Vienna Circle*. We find other reflections on the theme in *Philosophical Grammar* and the *Big Typescript* [18].

2.1 How probability works in the Tractatus

The notion of probability presented in Wittgenstein's *Tractatus Logico Philosophicus* may initially strike one as unusual, particularly when compared to the conventional modern perspective on probability. Wittgenstein's singular definition of probability is articulated in proposition 5.15:

> [5.15] If T_r is the number of the truth-grounds of a proposition r, and if T_{rs} is the number of the truth-grounds of a proposition s that are at the same time truth-grounds of r, then we call the ratio T_{rs}/T_r the degree of probability that the proposition r gives to the proposition s.

In practical terms, this approach entails examining instances where a belief proposition holds true and quantifying how many of these instances align with the true instances of the proposition under analysis, based on the original set of beliefs.

Example 2.1. Let us now examine an example drawn from everyday life: the act of tossing a coin. The main proposition under consideration will be denoted as $x \underline{\vee} y$, where the symbol $\underline{\vee}$ represents the mutually exclusive disjunction. In this context, the two possible outcomes, i.e., "head" and "tail," are mutually exclusive. The truth table for the proposition $x \underline{\vee} y$ is as follows:

	$x \underline{\vee} y$	x	y
1	F	T	T
2	**T**	**T**	F
3	**T**	F	**T**
4	F	F	F

Considering only the cases where $x \underline{\vee} y$ is true, we find that only the second and the third rows satisfy this condition. Now, let's determine the probabilities of x and y given the proposition $x \underline{\vee} y$. For the proposition x, out of the two instances where $x \underline{\vee} y$ is true, only the second instance has x as true while the third instance has x as false. Therefore, the probability of x given $x \underline{\vee} y$ is $1/2$. Similarly, for the proposition y, out of the two instances where $x \underline{\vee} y$ is true, only the third instance has y as true while the second instance has y as false. Thus, the probability of y given $x \underline{\vee} y$ is also $1/2$.

To summarize, when we toss a coin and consider the mutually exclusive disjunction proposition $x \underline{\vee} y$, the probability of x and y given this proposition is $1/2$ for both cases, as expected.

The reason behind Von Wright's perspective in [20] depicting the considered set as a set of beliefs becomes evident. Beliefs are perceived by an agent as unequivocally true, and this aligns with Wittgenstein's approach in the Tractatus. To provide further clarity, we shall introduce a novel formalization that was not presented by Wittgenstein, but that is very similar to the one used for conditionalization: this is due to the fact that Wittgenstein's method can be addressed as a formalization of conditionalization. In this formalization, we define the subscript as the set of beliefs under consideration for probability calculus. To illustrate this, let us revisit example 2.1, where the belief set considered was $x \underline{\vee} y$. Accordingly, the probability that x occurs, given $x \underline{\vee} y$, is denoted as $p_{x \underline{\vee} y}(x) = 1/2$.

Example 2.2. Let's now explore the case of the two coins problem. Imagine we need to toss two coins, and we want to calculate the probabilities using Wittgenstein's method. In this scenario, the proposition we'll analyze is $x \wedge r$, which, represents the probability of obtaining two heads from tossing two coins. To proceed, we need to consider each coin's results separately: let's designate the first coin's outcomes as x or y, and the second coin's outcomes as r or s.

The truth table for the proposition $(x \underline{\vee} y) \wedge (r \underline{\vee} s)$ is shown below:

	$(x \underline{\vee} y) \wedge (r \underline{\vee} s)$	x	y	r	s	$(x \wedge r)$	$(y \wedge s)$
1	F	T	T	T	T	T	T
2	F	T	T	T	F	T	F
3	F	T	T	F	T	F	T
4	F	T	T	F	F	F	F
5	F	T	F	T	T	T	F
6	**T**	**T**	F	**T**	F	**T**	F
7	**T**	**T**	F	F	**T**	F	F
8	F	T	F	F	F	F	F
9	F	F	T	T	T	F	T
10	**T**	F	**T**	**T**	F	F	F
11	**T**	F	**T**	F	**T**	F	**T**
12	F	F	T	F	F	F	F
13	F	F	F	T	T	F	F
14	F	F	F	T	F	F	F
15	F	F	F	F	T	F	F
16	F	F	F	F	F	F	F

Let's consider that $A = (x \underline{\vee} y) \wedge (r \underline{\vee} s)$. As per Wittgenstein's method, we find that $p_A(x) = p_A(y) = p_A(r) = p_A(s) = 0.5$, and $p_A(x \wedge r) = p_A(y \wedge s) = 0.25$. This confirms that the probability of getting two consecutive heads is 25%, which is the same for getting the first toss as head and the second toss as tail. Meanwhile, the probability of the first toss resulting in heads is 50%.

Let's now examine the likelihood of achieving a scenario with two heads, denoted as $x \wedge r$. Upon closer inspection, we observe that the condition $x \wedge y$ holds true solely in the sixth line, among a total of four instances where truth is affirmed. Similarly, this pattern emerges with $y \wedge s$, representing the chance of obtaining one head and one tail. This leads us to conclude that despite the method's seemingly unconventional nature, it harmoniously adheres to the principles governing probabilities.

2.2 Two elementary propositions

In one of the most significant statements concerning probability, one particular proposition stands out:

> [5.152] Two elementary propositions give one another the probability 1/2.

Von Wright [20, p. 262] argues that the interpretation of this statement depends on one's understanding of *elementary propositions.* However, in my view, now that the system is clarified, Wittgenstein's intended meaning becomes quite apparent. Let's consider a single elementary proposition, denoted as x, and an unrelated proposition, denoted as y. The corresponding truth table is as follows:

	x	y
1	**T**	**T**
2	**T**	F
3	F	T
4	F	F

As we observe from the truth table, y is true only once out of the two total instances. This arises from the fact that we have no information about the relationship between x and y; the only knowledge we possess is that they are not mutually related.

It is essential to emphasize that, in 5.152, Wittgenstein stated that *"Two independent propositions give one another the probability 1/2."* This statement, however, presents a particular problem due to the definition of independence provided in the same proposition in the Prototractatus [17] and in the first version of the Tractatus [1, 20]: *When propositions have no truth-arguments in common with one another, we call them independent of one another.* This implies that propositions like $x \vee \neg x$ and $y \vee \neg y$ are independent, but their probability is not 1/2; rather, it is 1 because both of them are tautologies. The same holds true for two contradictions, which can't be calculated because on the left we have an undetermined value. It is plausible that Wittgenstein was aware of this issue and thus modified the text in the second edition of Tractatus (1933) [1, 20].

3 Wittgenstein and supraclassical logic

Makinson's contribution that will be analysed here lies in his introduction of the concept of supraclassical logic, as documented in [12]. Supraclassical logics, a realm of formal reasoning that transcends the limitations of classical logic, have garnered substantial interest due to their capacity to deduce conclusions beyond what classical logic traditionally allows. Makinson employs a diverse array of methodologies to achieve this expansion, with one prominent approach involving the incorporation of sets of beliefs into the logical framework. Through this technique, propositions that typically remain undecidable within classical logic can be validated as true owing to the presence of the supplementary belief sets. Notably, this approach exhibits intriguing parallels with the philosophical underpinnings of Wittgenstein's work.

It's worth noting that while Wittgenstein's approach also explores the notion of probability, Makinson's focus in the initial section of the book centers on propositional logic, however, as the book progresses, Makinson delves into the realm of probability theory. It is from this exploration that the seeds of inspiration for our concept of linking these seemingly distant authors were sown.

The interesting thing of Wittgenstein's method is that was considered only a generalization of Laplace's principle of indifference: *if there aren't evidences that one outcome is more preferable than another, then the agent must distribute her credences equally among the total number of outcomes*. Actually it isn't just that, it is something more deep: Wittgenstein in fact proposed a method that considers one proposition as true and he compares that proposition with what we want to know. The number that he obtains is on one hand the generalization of Laplace principle, but on the other hand is proposing a different kind of logic, that is not only analyzing probability, but also compare a proposition with a given set of beliefs.

Let's consider example 2.2, depicting a coin toss. Notably, deriving $\vdash_B x \underline{\vee} y$ is elusive; its truth isn't immediately evident. However, Wittgenstein's view offers a fresh angle. It prompts us to assess not only binary truth but also the proposition's frequency amid all potential outcomes.

Wittgenstein introduces a novel stance on belief-logic dynamics. He positions probability as a broader extension of classical logic, though not a complete supraclassical logic. This viewpoint enriches our grasp of proposition nuances in various contexts. Wittgenstein's insight unveils the intricate ties between beliefs and logic. While probability isn't fully supraclassical, it links these domains, hinting at exciting avenues for future exploration and understanding.

3.1 Kolmogorov's axioms and Wittgenstein truth tables

Another very interesting point to note is that Wittgenstein's truth tables satisfy the Kolmogorov's axioms, i.e., the four Kolmogorov's axioms [12] are proved by the truth tables as intended in the considered part of the *Tractatus*. Informally they were firstly proved as provable in Wittgenstein's system in [13]. The axioms are as follows:

(K1) $0 \leq p(x) \leq 1$

(K2) $p(x) = 1$ for some formula x

(K3) $p(x) \leq p(y)$ whenever $x \vdash y$

(K4) $p(x \vee y) = p(x) + p(y)$ whenever $x \vdash \neg y$

(K1) and (K2) follow from construction: the final value must be a number between 0 and 1. It can't be less than 0 because the worst that can happen is that, as a belief, we have a contradiction, i.e., all instances are false. On the other hand, if the proposition we are analysing is equivalent to our belief or is a tautology, we will obtain the value of 1, but not more. This last consideration let the proof of (K2) obvious. (K3) can be proved thanks to the following truth table, where instead of $x \vdash y$, we consider $x \to y$ as true, that is a classical translation:

$K3$	$x \to y$	x	y
1	**T**	**T**	**T**
2	F	T	F
3	**T**	F	**T**
4	**T**	F	F

where $p_{x\to y}(x) = 1/3$ and $p_{x\to y}(y) = 2/3$, so $p_{x\to y}(x) \leq p_{x\to y}(y)$ and (K4) can be proved by the following:

$K4$	$x \to \neg y$	x	y	$x \vee y$
1	F	T	T	T
2	**T**	**T**	F	**T**
3	**T**	F	**T**	**T**
4	**T**	F	F	F

where $p_{x\to\neg y}(x) = 1/3$, $p_{x\to\neg y}(y) = 1/3$ and $p_{x\to\neg y}(x \vee y) = p_{x\to\neg y}(x) + p_{x\to\neg y}(y) = 1/3 + 1/3 = 2/3$ as wanted.

If we want to prove something generic the things become a little bit worse, because we have to check every case, for example if we want to prove

$(K5)$ $p(\neg x) = 1 - p(x)$ we must distinguish between the four combination of truthfulness and falsehood.

$K5$	$Formula$	x	$\neg x$
1	**T**	**T**	F
2	F	F	T

$p_{formula}(x) = 1$, $p_{formula}(\neg x) = 0$ and $p_{formula}(\neg x) = 1 - p_{formula}(x)$.

$K5$	$Formula$	x	$\neg x$
1	**T**	**T**	F
2	**T**	F	**T**

$p_{formula}(x) = 0.5$, $p_{formula}(\neg x) = 0.5$ and $p_{formula}(\neg x) = 1 - p_{formula}(x)$.

$K5$	$Formula$	x	$\neg x$
1	F	T	F
2	**T**	F	**T**

$p_{formula}(x) = 0$, $p_{formula}(\neg x) = 1$ and $p_{formula}(\neg x) = 1 - p_{formula}(x)$.

$K5$	$Formula$	x	$\neg x$
1	F	T	F
2	F	F	T

This last case is obviously special because we are giving a contradiction formula as a belief, so it's always false. Despite this, it was not really useful proving K5 from a formal point of view, because once K1-K4 were proved, than also K5 is provable from the first four axioms without using the truth tables.

Proving the Kolmogorov's axioms has a double benefit: it proves that Wittgenstein's idea of probability is something related to the common idea of it and it permits us to restrict the set of valuations to make a supraclassical logic.

3.2 Why Wittgenstein's probability is a supraclassical logic?

We have now shown that Wittgenstein's probability satisfies Kolmogorov's axioms. We can now turn to the main problem: is Wittgenstein's probability a supraclassical logic? Until now, we have only described a probabilistic logic, not a supraclassical one.

Definition 3.1 (Supraclassical logic)**.** A supraclassical logic is a logic that can derive more than classical logic usually permits, i.e., if $\vdash_S$ is the symbol for the supraclassical logic derivation, than it can be the case that $p \vdash_S q$ also if $p \nvdash q$.

In [12], Makinson elucidates the process of constructing a supraclassical logic thanks to three different techniques. The first of them, and the one that we will consider here, is to add a new set of axioms, namely beliefs, to let the logic prove more than classical logic usually permits. This method let the logic gain the ability to derive more than usual, also if it loses some property like the substitution, i.e., it loses Post Completeness.[1]

Following Wittgenstein's method it is clear why this is a supraclassical logic: it considers beliefs that are added into the system. Beliefs are exactly the left part of the derivation that we have made in the previous sections: beliefs for Wittgenstein, also if he doesn't call them this way, are the starting point to derive a certain number greater than 0, i.e., the value that classical logic would assign to the examples made earlier.

Someone can argue that Makinson, later in the book, integrates this with the probability theory, in particular he focuses on the non-monotonic version of the probabilistic supraclassical logic, so why can't we concentrate on them? The case that Makinson considers is useful if we want to create a non monotonic supraclassical logic, but this is not possible in the Wittgenstein's framework.

The Makinson's approach to supraclassical probabilistic logics involves the imposition of constraints on valuations. The crux of this approach lies in the selection of a specific subset, denoted as Q, extracted from the larger set P. Intriguingly, this subset Q possesses the unique property of assigning a probability value of 1 to a designated formula, even in scenarios where the encompassing set P fails to do so. To illustrate this, let us consider an example involving inconsistent sets of beliefs. Typically, an inconsistent set would attribute a probability value of 0 to every proposition. However, through the strategic confinement of the set to a consistent subset, the assignment of a probability value becomes viable.

This principle can be extended to various contexts. For instance, imagine the set P representing the logical conjunction $x \wedge y$, and we seek to ascertain the probability value of x. In the absence of constraints, the resultant probability would be 0.5. Nevertheless, by confining the analysis to solely the proposition x, we can derive a probability value of $p(x) = 1$.

[1]This result is proved independently by Makinson, saying that it loses substitution, but the original result was proved by Emil Post in his doctoral thesis, see [16].

The reason why we can't concentrate on this is that Wittgenstein's framework inherently lacks the capacity to accommodate non-monotonic reasoning. This limitation stems from the framework's heavy reliance on conditionalizations, a foundational aspect that precludes the integration of probabilistic non-monotonic logic. Makinson's development of a probabilistic non-monotonic logic necessitates the abandonment of the very concept of conditionalization, which, as the framework is originally constructed, proves unfeasible within this context.

In summation, Makinson's exploration of supraclassical logic creation through probability manipulation, as expounded in [12], can be effectively applied to Wittgenstein's methodology, in fact this way to treat probability is in fact a supraclassical logic. This methodology finds resonance even within Wittgenstein's theoretical paradigm. However, it is crucial to recognize that while Wittgenstein's framework is intrinsically tied to conditionalization, this feature inhibits the emergence of probabilistic non-monotonic logic, a frontier that Makinson's approach admirably advances by relinquishing the constraints of conditionalization.

4 Generalization of Wittgenstein's probability

We have shown that Wittgenstein's probability is consistent and it is a probabilistic logic; we have also shown that it is a supraclassical logic and this will help us for the following result: proving that thanks to Wittgenstein's method it is possible to solve a class of belief paradoxes, such as the Lottery Paradox [5, 8, 11] introduced in [10]. The lottery paradox is defined as follows:

> Let's consider a fair 1000-ticket lottery that has only one winning ticket. A perfectly rational agent knows that each ticket has a probability of 999/1000 of not winning. Thus, it is rational for the agent to accept that each ticket will not win because this probability is greater than her Lockean threshold. This reasoning can be extended to every other ticket in the lottery, leading to the conclusion that somehow every ticket will not be the winning ticket. However, the lottery is fair, so the conjunction of all these statements has to be false, rather than true as it appears.

The idea of solving this paradox thanks to Wittgenstein's idea is interesting because of the following proposition, that we also have addressed in the introduction:

> [5.156] It is in this way that probability is a generalisation. It involves a general description of a propositional form. We use probability only in default of certainty - if our knowledge of a fact is not indeed complete, but we do know something about its form. (A proposition may well be an incomplete picture of a certain situation, but it is always a complete picture of something.) A probability proposition is a sort of excerpt from other propositions.

In fact, if we follow Wittgenstein's idea, we can solve the Lottery paradox inside a generalisation of classical logic. The proof of the fact that Wittgenstein's method is a supraclassical logic, also coheres with this proposition, creating a link between a 1929's method and a very recent one. This solution can be given without dropping the principle of conjunction between rational beliefs as the author of the paradox, Kyburg, has originally suggested in [10].

One of the intriguing outcomes facilitated by this approach is its capacity to address belief paradoxes, such as the Lottery Paradox. This paradox has generated substantial literature and is readily demonstrable that, under a classical framework and within the context of the Lockean Thesis, it remains paradoxical. However, if we adopt Wittgenstein's perspective that this form of probability extends classical logic, we can identify a method within it that effectively resolves the Lottery Paradox.

To achieve this, we need to establish that when dealing with a conjunction involving a finite yet arbitrarily large number of elementary propositions, where all but one are negative, only a singular True line emerges. Furthermore, it becomes essential to demonstrate that this true line occupies a specific position within the matrix and maintains its uniqueness as we select distinct propositions, each with the positive formula in a different position. Utilizing these insights, we can construct a disjunction encompassing all conceivable scenarios, resulting in exactly n True lines—where n represents the count of literals within the formula. Firstly let's see the following true table, but let's consider that instead of including both T (True) and F (False) values for each proposition, we have opted for a more readable table format. This is why, in our table, $\neg p_1$ is represented as F in its initial entry:

	$\neg p_1$	$\wedge$	$\neg p_2$	$\wedge$	$\dots$	$\wedge$	p_x	$\wedge$	$\neg p_{x+1}$	$\wedge$	$\dots$	$\wedge$	$\neg p_n$
1	F		F				T		F				F
2	F		F				T		F				T
$\vdots$	$\vdots$		$\vdots$				$\vdots$		$\vdots$				$\vdots$
2^{n-1}	F		T				T		T				T
$2^{n-1}+1$	T		F				F		F				F
$\vdots$	$\vdots$		$\vdots$				$\vdots$		$\vdots$				$\vdots$
$2^{n-1}+2^{n-2}$	T		F				T		F				T
$2^{n-1}+2^{n-2}+1$	T		T				F		T				F
$\vdots$	$\vdots$		$\vdots$				$\vdots$		$\vdots$				$\vdots$
$2^n-2^{n-x}-1$	T		T				T		T				F
2^n-2^{n-x}	**T**		**T**				**T**		**T**				**T**
$2^n-2^{n-x}+1$	T		T				F		F				F
$\vdots$	$\vdots$		$\vdots$				$\vdots$		$\vdots$				$\vdots$
2^n	T		T				F		F				T

This truth table needs some hint to let it be cleared:

1. To enhance clarity in tracking transitions from T to F, we have explicitly highlighted the most significant changes. For example, 2^{n-1} represents the last row where the truth value of $\neg p_1$ changes, occurring exactly at the midpoint of the entire truth table. Similarly, $2^{n-1}+2^{n-2}$ marks the last row before the intermediate change of $\neg p_2$.

2. The most intriguing row in the table is $2^n - 2^{n-x}$ because it consists entirely of T instances. This is a result of the fact that on the left side of p_x, we only have T instances that continually double in number with each iteration. On the right side, we observe a similar pattern, but with 'T' instances halving until we reach the single T instance for $\neg p_n$.

3. The value of 2^n-2^{n-x} corresponds to the last row before the truth value of p_x changes. It can also be expressed as $\sum_{i=1}^{x} 2^{n-i}$ as it requires summing the halved values successively, reflecting the decreasing number of T instances with each new proposition considered.

The following theorem is the main theorem to be proved in order to generalize Wittgenstein's probability:

Theorem 4.1. *If a proposition made by an arbitrary number of elementary letters is made by all negated formulas and one positive formula, the only line that is made by true instances is the line marked with the number $2^n - 2^{n-x}$, where x is the position of the elementary letter starting from the left.*

We will first explain the process by which this truth table was created before providing the proof: intuitively, to find which line is true we have to consider the two extreme cases, i.e., $p_1 \wedge \cdots \wedge \neg p_n$ and $\neg p_1 \wedge \cdots \wedge p_n$; then we have to prove it for a generic p_x between p_1 and p_n. Let's consider then the following where the positive letter is the first, i.e., p_1:

$$
\begin{array}{c|ccccccccc}
 & p_1 & \wedge & \neg p_2 & \wedge & \neg p_3 & \wedge & \dots & \wedge & \neg p_n \\
1 & \mathbf{T} & & F & & F & & & & F \\
2 & \mathbf{T} & & F & & F & & & & T \\
\vdots & \vdots & & \vdots & & \vdots & & & & \vdots \\
2^{n-2} & \mathbf{T} & & F & & T & & & & T \\
2^{n-2}+1 & \mathbf{T} & & \mathbf{T} & & F & & & & F \\
\vdots & \vdots & & \vdots & & \vdots & & & & \vdots \\
2^{n-2}+2^{n-3} & \mathbf{T} & & \mathbf{T} & & F & & & & F \\
2^{n-2}+2^{n-3}+1 & \mathbf{T} & & \mathbf{T} & & \mathbf{T} & & & & T \\
\vdots & \vdots & & \vdots & & \vdots & & & & \vdots \\
2^{n-1} & \mathbf{T} & & \mathbf{T} & & \mathbf{T} & & & & \mathbf{T} \\
2^{n-1}+1 & F & & F & & F & & & & F \\
\vdots & \vdots & & \vdots & & \vdots & & & & \vdots \\
2^n & F & & T & & T & & & & T
\end{array}
$$

As observed, to the right of a positive propositional letter, we notice a diminishing count of admissible lines. The truth line is in fact 2^{n-1} that is exactly $\sum_{i=1}^{x} 2^{n-i} = 2^n - 2^{n-x}$ where $x = 1$, following that $2^n - 2^{n-1} = 2^{n-1}$. This phenomenon arises because each time the upper half consists solely of false instances and because of the conjunction property, it is possible to consider only the bottom half each time. This pattern persists until we reach the final propositional letter, which renders only one line true among the total of 2^n lines.

On the other hand, if we consider the other limit case, considering that the only positive formula is p_n we obtain:

	$\neg p_1$	$\wedge$	$\neg p_2$	$\wedge$	$\neg p_3$	$\wedge$	$\dots$	$\wedge$	p_n
1	F		F		F				T
2	F		F		F				F
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
2^{n-1}	F		T		T				F
$2^{n-1}+1$	**T**		F		F				T
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
$2^{n-1}+2^{n-2}$	**T**		F		T				F
$2^{n-1}+2^{n-2}+1$	**T**		**T**		F				T
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
$2^{n-1}+2^{n-3}$	**T**		**T**		F				F
$2^{n-1}+2^{n-3}+1$	**T**		**T**		**T**				T
$\vdots$	$\vdots$		$\vdots$		$\vdots$				$\vdots$
2^n-1	**T**		**T**		**T**				**T**
2^n	T		T		T				F

This implies that to the left of p_n, we witness a diminishing set of potential truth instances, as previously explained, with only the lower half being considered for conjunction. Ultimately, the penultimate line stands as the sole truth-bearing one, i.e. $2^n - 1$ that satisfies the formula $2^n - 2^{n-x}$, where $x = n$: $2^n - 2^{n-n} = 2^n - 1$.

For any generic positive value of p_x between p_1 and p_n, we need to consider a range of values between these two extremes. If we examine the first truth table, we can observe that when we start with $\neg p_1$, we only need to focus on the second half of the truth values since the false instances in the first half are not relevant. Moving on to $\neg p_2$, we continue to concentrate on the second half, and this pattern continues until we reach $\neg p_{x-1}$.

When we consider p_x as true, the order of the true values changes. In other words, the first half of the remaining true instances now becomes false, while the second half remains true. As we proceed to $\neg p_{x+1}$, the true instances again halve in the lower part, and this process continues until we arrive at $\neg p_n$, which represents the last line where truth is possible. In conclusion, the exact line where p_x is true in the table corresponds to $2^n - 2^{n-x}$.

Combining these outcomes elucidates why precisely the line $2^n - 2^{n-x}$ is replete with truth instances while the others cannot be true. Furthermore, this unique truth-bearing line varies for each distinct variable x, as evidenced by the changing values of $2^n - 2^{n-x}$. From these considerations it is easy to understand why the line made only by true instances is the last line where p_x has a T-value: the idea of proof lies on this fact.

Proof of Theorem 4.1. We can prove Theorem 4.1 by induction, leveraging the fact that $2^n - 2^{n-x}$ represents the last line where p_x has a truth value of T. The idea is to establish by induction that $2^{n+1} - 2^{n+1-x}$ remains the last line where p_x has a truth value of T and that each other propositional letters have a T value in that line when we have $n+1$ propositional letters.

Base case: For the base case when $n = 1$, we note that the only true line is the first one. This can be verified by calculating $2^1 - 2^0 = 2 - 1 = 1$, which matches the truth value in the first line.

Inductive step: Now, let's consider the inductive step. Assuming that the line number $2^n - 2^{n-x}$ has only T instances for some value of n, we aim to show that if x remains the same, then the new line should be twice the value of $2^n - 2^{n-x}$.

This is because of the construction of a truth table: if a line for a certain propositional letter, let's say line number i for letter a, was labeled as T (F) in a truth table created for n elementary propositions, then line $2i$ for letter a will also be labeled as T (F) when adding a new elementary letter.

Proving this implies that at line $2(2^n - 2^{n-x})$ for $n+1$ elementary letters, each propositional letter between 1 and n will be true. Moreover, the new elementary letter, $\neg p_{n+1}$, will be true in that line because it alternates between F and T (initially F because $\neg p_{n+1}$ is false in the first line, being a negated formula). This means that the T instances will appear on even lines, and $2(2^n - 2^{n-x})$ is even, completing the correspondence between the two truth tables.

To complete the proof, we need to establish a correspondence between $2(2^n - 2^{n-x})$ and $2^{n+1} - 2^{n+1-x}$. We can easily demonstrate that:

$$2^{n+1} - 2^{n+1-x} = 2(2^n - 2^{n-x})$$

This equation establishes the desired relationship between the new line and the previous one, confirming that it aligns with our expectations. As the inductive hypothesis establishes, the line $2^n - 2^{n-x}$ was true for n. Therefore, the line $2(2^n - 2^{n-x})$ will also be true for $n+1$ because the number of lines doubles, concluding the proof.

$\square$

Going back to the initial problem of the Lottery paradox: due to the uniqueness of each value of x on every occasion the disjunction of various conjunctions, where each conjunction follows a pattern of positive and negative literals, specifically:

$$
\begin{aligned}
&(p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n) \vee \\
&\qquad\qquad \cdots \vee (\neg p_1 \wedge \cdots \wedge p_x \wedge \cdots \wedge \neg p_n) \vee \\
&\qquad\qquad\qquad\qquad \cdots \vee (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge p_n)
\end{aligned}
$$

will have exactly n lines. This is due to the fact that we want to formalize a lottery and this means that we want the exclusive disjunction for each ticket.

Remarkably, this composite expression will consistently exhibit precisely n instances of truth lines across all possible configurations of truth values. It is noteworthy that our previous investigation has conclusively established the singularity of the line characterized by a T-value. This uniqueness materializes as $2^n - 2^x$ for values of x ranging from 1 to n, with each individual value of x generating a distinct outcome. We can see it in the following table:

	$(p_1 \wedge \cdots \wedge \neg p_n)$	$\vee$	$\dots$	$\vee$	$(\neg p_1 \wedge \cdots \wedge p_n)$	
1	F				F	F
$\dots$						
2^{n-1}	**T**				F	**T**
$\dots$						
$2^n - 1$	F				**T**	**T**
2^n	F				F	F

In total we have n true instances and this means that when we consider only one proposition, such as $(p_1 \wedge \cdots \wedge \neg p_n)$, it will be true $1/n$ times.

Example 4.1. Let's see an example: let's consider that the lottery has 1000 tickets, than the proposition will be $A = (p_1 \wedge \cdots \wedge \neg p_{1000}) \vee \cdots \vee (\neg p_1 \wedge \cdots \wedge p_{1000})$ and let's say that the ticket that we have bought is the ticket number 543, then we have to compare A with $\neg p_1 \wedge \cdots \wedge p_{543} \wedge \cdots \wedge \neg p_{1000}$. The only true line for $\neg p_1 \wedge \cdots \wedge p_{543} \wedge \cdots \wedge \neg p_{1000}$ will be $2^{1000} - 2^{1000-543} = 2^{1000} - 2^{457}$. This line will be one of the 1000 true lines of the proposition A for construction and this means that the final probabilistic value of the truthfullness of $\neg p_1 \wedge \cdots \wedge p_{543} \wedge \cdots \wedge \neg p_{1000}$ given A will be $1/1000$.

5 Conclusions

We have addressed a multitude of challenges within Wittgenstein's probabilistic framework through the comprehensive analysis presented in this paper. Wittgenstein's initial perspective on probability may, at first glance, appear unconventional, yet it takes on a distinct character when we emphasize its inherent consistency. Our research demonstrates that Wittgenstein's approach adheres faithfully to Kolmogorov's axioms and qualifies as a supraclassical logic. Building upon these fundamental insights, we have solved the Lottery Paradox, which, within this framework, ceases to be paradoxical and instead finds resolution through an extension of classical logic.

While this methodology does not represent an entirely revolutionary paradigm shift, we believe it has received less attention than its merits warrant. The innovative incorporation of beliefs as a foundational element in the analysis of probability introduces a novel dimension to the field. Looking ahead, we are optimistic that our exploration of supraclassical logic and probabilistic reasoning will make valuable contributions towards the development of a new, robust supraclassical probabilistic logic. This emerging framework, enriched by the incorporation of Wittgenstein's philosophical insights, has the potential to establish its own solid foundations within the realms of both logic and philosophy.

References

[1] Cuffaro, Michael E. "Wittgenstein on Prior Probabilities". In: *Proceedings of the Canadian Society for History and Philosophy of Mathematics* 23 (2010), pp. 85–98.

[2] Figueiredo, Florian Franken. "Wittgenstein's 1929–30 inquiries into probability". In: *Philosophical Investigations* 46.1 (2023), pp. 50–75.

[3] Finetti, Bruno De. *La logica dell'incerto*. Libreria Editrice Francesco Perrella, 1931.

[4] Finetti, Bruno De. *La logica dell'incerto*. Il Saggiatore, 1989.

[5] Foley, Richard. "The Epistemology of Belief and the Epistemology of Degrees of Belief". In: *American Philosophical Quarterly* 29.2 (1992), pp. 111–124.

[6] Hajek, Alan. "Interpretations of Probability". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2019. Metaphysics Research Lab, Stanford University, 2019.

[7] Halpern, Joseph Y. *Reasoning about Uncertainty*. Cambridge, MA, USA: MIT Press, 2003.

[8] Hawthorne, James. "The Lockean Thesis and the Logic of Belief". In: *Degrees of Belief*. Ed. by Franz Huber and Christoph Schmidt-Petri. Synthese Library: Springer, 2009, pp. 49–74.

[9] Hay, Clare. "Probability in Wittgenstein's Tractatus". In: *Philosophical Investigations* 45.2 (2022), pp. 130–145.

[10] Kyburg, Henry Ely. *Probability and the Logic of Rational Belief*. Middletown, CT, USA: Wesleyan University Press, 1961.

[11] Leitgeb, Hannes. *The Stability of Belief: How Rational Belief Coheres with Probability*. Oxford University Press, 2017.

[12] Makinson, D. *Bridges from classical to nonmonotonic logic*. Lightning Source, Milton Keynes, 2005.

[13] Ongaro, Malvina. "The Interpretation of Probability in the Tractatus Logico-Philosophicus". In: *Wittgenstein-Studien* 12.1 (2021), pp. 131–144.

[14] Paris, J. B. *The Uncertain Reasoner's Companion: A Mathematical Perspective*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1995.

[15] Piazza, M. and Pulcini, G. "Uniqueness of axiomatic extensions of cut-free classical propositional logic". In: *Logic Journal of the IGPL* 24.5 (Oct. 2016), pp. 708–718.

[16] Post, Emil. "Introduction to a general theory of elementary propositions". In: *From Frege to Gödel: A Source Book in Mathematical Logic*. Ed. by Jean vana Heijenoort. Harvard University Press, 1990.

[17] Wittgenstein, Ludwig. *Prototractatus an Early Version of Tractatus Logico-Philosophicus. Edited by B.F. Mcguinness, T. Nyberg and G.H. Von Wright, with a Translation by D.F. Pears and B.F. Mcguinness. An Historical Introd. By G.H. Von Wright and a Facsim. Of the Author's Manuscript*. Cornell University Press, 1971.

[18] Wittgenstein, Ludwig. *The Big Typescript: TS 213, German English Scholars' Edition*. Blackwell Pub; Bilingual, 2012.

[19] Wittgenstein, Ludwig. *Tractatus Logico-Philosophicus*. Routledge and Kegan Paul, 1922.

[20] Wright, G. H. Von. "Wittgenstein's views on probability". In: *Revue Internationale de Philosophie* 23.88/89 (2/3) (1969), pp. 259–279.