Inside and outside the boundaries: probability in Wittgenstein's Tractatus

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Abstract

The relationship between probability and classical logic can be approached from various angles. While the prevailing perspective often views probability as an extension of classical logic, there exists a less conventional approach that involves interpreting probability within the framework of classical logic itself. This alternative viewpoint, though less common, holds considerable interest and is exemplified in the works of philosophers such as Wittgenstein, De Finetti, Makinson, among others. Here, our focus lies on Wittgenstein's contribution, which holds both historical and philosophical significance in bridging probability and classical logic.

In his Tractatus, Wittgenstein introduced a method for computing probability using truth tables, which subsequently influenced the work of scholars like Carnap and Ramsey. Despite its historical importance, Wittgenstein's method has often been overlooked in the literature. Some scholars have interpreted it as an extension of the indifference principle, while others have seen it as an exploration of the relationship between beliefs and logic. Wittgenstein's method involves comparing two propositions: one analyzed solely in instances of truth, while the other is considered only when the first holds true. Remarkably, this approach bears resemblance to Makinson's supraclassical logic, albeit with differing methodologies.

This study aims to clarify Wittgenstein's method and its connection to probability and classical logic, with a particular focus on resolving the Lottery Paradox within the framework established by Wittgenstein.

1 Introduction

In a separate publication (Bizzarri 2024), we endeavored to offer a thorough analysis of Wittgenstein's concept of probability, demonstrating how it resolves or, better, dissolves the lottery paradox within his framework. Here, we aim to delve into the philosophical aspect of the coherence between probability and classical logic, starting from probability as presented in Wittgenstein's *Tractatus*.

Wittgenstein defines probability in terms of the relationship between "belief's truth-possibilities" (*Wahrheitsmöglichkeiten*) (Figueiredo 2023; Hay 2022; Cuffaro 2010; Ongaro 2021) and the truth possibilities of the proposition under consideration. Throughout his Tractatus (Wittgenstein 1922), Wittgenstein asserts that probability is inherently *a priori*, a stance he maintains in his later works where he vehemently rejects *frequentism* as the correct interpretation of probability.

> Let's assume that someone playing dice every day were to throw, say, nothing but ones for a whole week, and that he does this with dice that turn out to be good when subjected to all other methods of testing, and that also produce the normal results when someone else throws them. Does he now have reason to assume a natural law here, according to which he always has to throw ones? Does he have reason to believe that things will continue in this way – or (rather) to assume that this regularity won't last much longer? So does he have reason to quit the game since it has turned out that he can throw only ones; or to continue playing, because now it is just all the more likely that on the next try he'll throw a higher number? – In actual fact he'll refuse to acknowledge the regularity as a law of nature; at least it will have to last for a long time before he'll consider this view of regularity. But why? – I think it's because so much of his previous experience in life refutes such a law, experience that has to be, so to speak – vanquished before we accept a totally new way of looking at things. (Wittgenstein 2012, 104e)

The philosophical idea of this paper lies on the fact that for Wittgenstein probability is a sort of extension of classical logic:

> [5.156] It is in this way that probability is a generalization.
> It involves a general description of a propositional form.
> We use probability only in default of certainty—if our knowledge

> of a fact is not indeed complete, but we do know something about its form.
> (A proposition may well be an incomplete picture of a certain situation, but it is always a complete picture of something.)
> A probability proposition is a sort of excerpt from other propositions.

In Wittgenstein's conception of probability, truth is not solely dictated by logic but also by *knowledge*, specifically beliefs. Consequently, propositions that don't conform to classical logic as tautologies can still be ascribed non-zero values within a probabilistic framework.

Furthermore, we establish a correlation between the well-known Lottery Paradox and Wittgenstein's concept of probability, showcasing its straightforward resolution within this framework while also presenting intriguing philosophical implications. By delving into these interconnections, our objective is to elucidate the distinctive characteristics and ramifications of Wittgenstein's probabilistic approach.

In the following two sections, we will revisit the concepts introduced in (Bizzarri 2024), omitting the details covered in the referenced paper. Additionally, in the third paragraph, we will present a philosophical argument that stems from probability in Wittgenstein's Tractatus and extends to the relationship between probability and Classical Logic.

2 Probability in the Tractatus

Wittgenstein's early notions regarding probability were first deliberated within the Circle of Vienna (Wright 1969) before undergoing refinement and solidification in the *Tractatus Logico-Philosophicus.* The treatment of probability in the *Tractatus* might seem peculiar at first glance, especially when contrasted with the conventional contemporary understanding of probability. Wittgenstein's distinct characterization of probability is elucidated in proposition 5.15:

> [5.15] If T_r is the number of the truth-grounds of a proposition r, and if T_{rs} is the number of the truth-grounds of a proposition s that are at the same time truth-grounds of r, then we call the ratio T_{rs}/T_r the degree of probability that the proposition r gives to the proposition s.

To understand better let's consider an example:

Example 2.1. Now, let's explore a common example from everyday life: flipping a coin. The central proposition we'll focus on is denoted as $x \underline{\vee} y$, where $\underline{\vee}$ signifies the mutually exclusive disjunction. In this scenario, the two potential outcomes, "heads" and "tails," are mutually exclusive. The truth table for the proposition $x \underline{\vee} y$ is as follows:

	$x \underline{\vee} y$	x	y
1	F	T	T
2	**T**	**T**	F
3	**T**	F	**T**
4	F	F	F

When considering only the instances where $x \underline{\vee} y$ holds true, we observe that only the second and third rows meet this criterion. Let's now calculate the probabilities of x and y given the proposition $x \underline{\vee} y$. For proposition x, among the two instances where $x \underline{\vee} y$ is true, only the second instance has x as true, while the third instance has x as false. Consequently, the probability of x given $x \underline{\vee} y$ is $1/2$. Similarly, for proposition y, among the two instances where $x \underline{\vee} y$ is true, only the third instance has y as true, whereas the second instance has y as false. Thus, the probability of y given $x \underline{\vee} y$ is also $1/2$.

In summary, when flipping a coin and considering the mutually exclusive disjunction proposition $x \underline{\vee} y$, the probabilities of x and y given this proposition are both $1/2$, as anticipated.

2.1 Kolmogorov's axioms and Wittgenstein truth tables

Wittgenstein's truth tables satisfy Kolmogorov's axioms, validated in the *Tractatus*. The axioms, informally established in previous work, are:

(K1) $0 \leq p(x) \leq 1$

(K2) $p(x) = 1$ for some formula x

(K3) $p(x) \leq p(y)$ whenever $x \vdash y$

(K4) $p(x \vee y) = p(x) + p(y)$ whenever $x \vdash \neg y$

(K1) and (K2) derive from construction, bounded between 0 and 1. (K3) is validated via a truth table, substituting $x \vdash y$ with $x \rightarrow y$ as true.

$K3$	$x \rightarrow y$	x	y
1	**T**	**T**	**T**
2	F	T	F
3	**T**	F	**T**
4	**T**	F	F

where $p_{x \to y}(x) = 1/3$ and $p_{x \to y}(y) = 2/3$, so $p_{x \to y}(x) \leq p_{x \to y}(y)$ and (K4) can be proved by the following:

$K4$	$x \to \neg y$	x	y	$x \vee y$
1	F	T	T	T
2	**T**	**T**	F	**T**
3	**T**	F	**T**	**T**
4	**T**	F	F	F

where $p_{x \to \neg y}(x) = 1/3$, $p_{x \to \neg y}(y) = 1/3$ and $p_{x \to \neg y}(x \vee y) = p_{x \to \neg y}(x) + p_{x \to \neg y}(y) = 1/3 + 1/3 = 2/3$ as wanted.

If we want to prove something generic the things become a little bit worse, because we have to check every case, for example if we want to prove $(K5)$ $p(\neg x) = 1 - p(x)$ we must distinguish between the four combination of truthfulness and falsehood.

$K5$	$Formula$	x	$\neg x$
1	**T**	**T**	F
2	F	F	T

$p_{formula}(x) = 1$, $p_{formula}(\neg x) = 0$ and $p_{formula}(\neg x) = 1 - p_{formula}(x)$.

$K5$	$Formula$	x	$\neg x$
1	**T**	**T**	F
2	**T**	F	**T**

$p_{formula}(x) = 0.5$, $p_{formula}(\neg x) = 0.5$ and $p_{formula}(\neg x) = 1 - p_{formula}(x)$.

$K5$	$Formula$	x	$\neg x$
1	F	T	F
2	**T**	F	**T**

$p_{formula}(x) = 0$, $p_{formula}(\neg x) = 1$ and $p_{formula}(\neg x) = 1 - p_{formula}(x)$.

$K5$	$Formula$	x	$\neg x$
1	F	T	F
2	F	F	T

This last case is obviously special because we are giving a contradiction formula as a belief, so it's always false. Despite this, it was not really useful proving K5 from a formal point of view, because once K1-K4 were proved, than also K5 is provable from the first four axioms without using the truth tables.

Proving the Kolmogorov's axioms has a double benefit: it proves that Wittgenstein's idea of probability is something related to the common idea of it and it permits us to restrict the set of valuations to make a supraclassical logic.

3 Generalization of Wittgenstein's probability

Wittgenstein's probability offers a consistent probabilistic logic within classical limits, enabling resolution of belief paradoxes like the Lottery Paradox. By extending classical logic, we maintain conjunction principles, contrary to previous suggestions. This approach addresses paradoxes effectively, notably the Lottery Paradox (Hawthorne 2009; Foley 1992; Leitgeb 2017; Kyburg 1961), which persists under classical frameworks. The Lottery Paradox is formulated as follows:

> Let's consider a fair 1000-ticket lottery that has only one winning ticket. A perfectly rational agent knows that each ticket has a probability of 999/1000 of not winning. Thus, it is rational for the agent to accept that each ticket will not win because this probability is greater than her Lockean threshold. This reasoning can be extended to every other ticket in the lottery, leading to the conclusion that somehow every ticket will not be the winning ticket. However, the lottery is fair, so the conjunction of all these statements has to be false, rather than true as it appears.

The idea of solving this paradox thanks to Wittgenstein's idea is interesting because of the following proposition, that we also have addressed in the introduction:

> [5.156] It is in this way that probability is a generalisation. It involves a general description of a propositional form. We use probability only in default of certainty - if our knowledge of a fact is not indeed complete, but we do know something about its form. (A proposition may well be an incomplete picture of a certain situation, but it is always a complete picture of something.) A probability proposition is a sort of excerpt from other propositions.

Leveraging Wittgenstein's notion of probability as a generalization, we demonstrate a method to resolve the paradox. This involves establishing a unique True line amidst a conjunction of numerous negative propositions, maintaining its position as propositions vary. Utilizing this insight, we construct a disjunction to encompass all scenarios, yielding exactly n True lines, where n is the count of literals within the formula

	$\neg p_1 \quad \wedge$	$\neg p_2 \quad \wedge \quad \ldots \quad \wedge$	$p_x \quad \wedge$	$\neg p_{x+1} \quad \wedge \quad \ldots \quad \wedge$	$\neg p_n$
1	F	F	T	F	F
2	F	F	T	F	T
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{n-1}	F	T	T	T	T
$2^{n-1}+1$	T	F	F	F	F
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2^{n-1}+2^{n-2}$	T	F	T	F	T
$2^{n-1}+2^{n-2}+1$	T	T	F	T	F
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2^n-2^{n-x}-1$	T	T	T	T	F
2^n-2^{n-x}	**T**	**T**	**T**	**T**	**T**
$2^n-2^{n-x}+1$	T	T	F	F	F
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^n	T	T	F	F	T

This truth table needs some hint to let it be cleared:

- Highlighting significant transitions enhances clarity. For instance, 2^{n-1} marks the last row where $\neg p_1$ changes, at the midpoint of the table. Similarly, $2^{n-1}+2^{n-2}$ precedes the change of $\neg p_2$.
- The row $2^n - 2^{n-x}$ is notable, filled entirely with T. It results from doubling T instances left of p_x while halving them right of p_x, ending with a single T for $\neg p_n$.
- 2^n-2^{n-x} signifies the last row before p_x changes, equivalent to $\sum_{i=1}^{x} 2^{n-i}$, summing halved values successively, reflecting diminishing T instances.

The following theorem is the main theorem to be proved in order to generalize Wittgenstein's probability:

Theorem 3.1. *If a proposition made by an arbitrary number of elementary letters is made by all negated formulas and one positive formula, the only line* *that is made by true instances is the line marked with the number* $2^n - 2^{n-x}$*, where* x *is the position of the elementary letter starting from the left.*

Proof. See (Bizzarri 2024) □

Thanks to Theorem 3.1, we can observe how the truth table regarding the Lottery Paradox can be resolved. Interestingly, a similar outcome was presented in Bizzarri 2024, albeit through a completely different method. However, we will delve into this in the next section. Surprisingly, the Lottery Paradox can also be solved in Fractional Semantics as presented in (Bizzarri 2023) and also in the limits of Classical Logic.

4 Probability and classical logic

Probability has long presented a challenging relationship with classical logic. On one hand, it appears to extend classical logic's, yet on the other hand, it seems to impose constraints on its rules. Within Wittgenstein's conceptual framework, probability finds a place within the limits of classical logic, particularly in his exploration of the interplay between beliefs and propositions.

By the way Wittgenstein's method was seen as a generalization over possibilities, also if the question is more challenging. In De Finetti's "theories of probabilities" (Finetti 1931), a clear distinction is drawn between possibilities, which are objective, and probabilities, which are subjective. Wittgenstein's perspective occupies a middle ground between these concepts. While he meticulously analyzes each possibility of falsity and truthfulness akin to De Finetti's framework, the subjective element emerges from the agent's capacity to select the initial set of propositions, intertwined with the agent's personal comprehension of a given argument.

Despite its significance, probability in Wittgenstein's oeuvre is often treated as peripheral, with scant exploration of his specific viewpoints on the subject. Notably, Wittgenstein's fundamental musings on the nature of probability are encapsulated in the Tractatus, commencing from proposition 5.1:

> [5.1] Truth-functions can be arranged in series. That is the foundation of the theory of probability

In essence, Wittgenstein's exploration of probability can be interpreted as a compelling endeavor to bridge the gap between beliefs and propositions, but also as the first tentative of include probability into the limits and confines of Classical Logic. While it shares similarities with objective possibilities, it also exhibits subjective features by empowering agents to shape the initial set of propositions according to their individual knowledge. Despite its relatively limited exposition, Wittgenstein's reflections on probability in the Tractatus offer invaluable insights into this intricate domain. Initial reflections on these can be traced back to the Notebooks 1914-1916 and discussions within the Vienna Circle.

In this argument, I posit that Wittgenstein's notion of probability, despite facing substantial critique—many of which have been aptly addressed by Cuffaro in (Cuffaro 2010)—remains a significant exemplar of the symbiotic relationship between classical logic and probability theory. This synergy has been further advanced by eminent philosophers such as Ramsey and De Finetti, who have embraced and expanded upon this conceptual interplay in their respective works. For instance, De Finetti elucidates in "Theories of Probabilities" that probability inherently resides within subjective realms, encapsulating one's "degree of beliefs." This intrinsic link between classical logic and the subjective assessment of probabilities is widely acknowledged and appreciated within philosophical discourse.

Moreover, what Wittgenstein suggested in the *Tractatus*, i.e., that probability is a relationship between beliefs and the logic, will be a firm point also in his later writings. For example in the Big Typescript he writes:

> 33.3 Die Induktion ist ein Vorgang nach einem ökonomischen Prinzip. [Induction is a process based on an economic principle.]Wittgenstein 2012

Articulating a notion that resonated strongly with De Finetti, it becomes evident that probability is inherently grounded in subjective interpretation and operates on an economic principle. By leveraging beliefs alongside classical logic, the framework fundamentally aligns itself with Classical Logic, thus situating probability within the confines of Classical Logic and concurrently diminishing its boundaries due to Post-Completeness.[1]

Concluding, Wittgenstein's view on probability has several peculiar aspects. If we follow Wittgenstein's idea until the very end, his view on probability remains within the boundaries of Classical Logic (it is, in fact, only a *generalization*, but the structure remained the same), and, as we have shown, it also satisfies Kolmogorov's axioms and resolves the Lottery Paradox. These significant aspects aid in understanding how probability can be constructed within or outside Classical Logic. Expanding the boundaries

[1]This perennial issue arises when beliefs are contextualized within Classical Logic, necessitating a trade-off between consistency and structural integrity. The forfeiture of structural integrity precludes the utilization of Substitution, a fundamental operation within classical logic.

of Classical Logic is technically challenging but straightforward: it suffices to add semantics that can reconnect our logic to the mathematical form of probability. Conversely, staying within the boundaries of Classical Logic is more difficult to justify but technically simpler and philosophically more intriguing. We believe that Wittgenstein was able to grasp many of the problems that logicians still face today when dealing with Probability and Classical Logic, and he resolved them in an elegant and synthetic manner. We propose that this initial attempt served as the foundation upon which De Finetti and Ramsey based their work, and its philosophical significance must be revitalized.

5 Conclusions

In this paper we provided a description of the first Wittgenstein's view on probability.

In our paper, we have extensively tackled the challenges posed by Wittgenstein's probabilistic framework, particularly focusing on the Lottery Paradox. At first glance, Wittgenstein's approach to probability may seem unorthodox, but upon closer examination, it reveals a coherent structure that aligns with Kolmogorov's axioms and qualifies as a supraclassical logic.

Our research underscores the consistency of Wittgenstein's perspective, offering a resolution to the Lottery Paradox within this framework. What was once considered a paradox now finds clarity through an extension of classical logic.

While our methodology isn't a complete departure from conventional approaches, it deserves more attention for its innovative incorporation of beliefs into the analysis of probability. This inclusion adds a fresh dimension to the field and sets the stage for the development of a robust supraclassical probabilistic logic.

Looking forward, we anticipate that our exploration of supraclassical logic and probabilistic reasoning, enriched by Wittgenstein's philosophical insights, will contribute significantly to the establishment of a solid foundation bridging logic and philosophy.

References

Bizzarri, Matteo (2023). "A solution to the Lottery Paradox through Fractional Semantics". In: *Fifth Pisa Colloquium in Logic, Language and Epis-*

temology. Ed. by Giacomo Turbanti Luca Bellotti Enrico Moriconi. Edizioni ETS, pp. 15–29.

Bizzarri, Matteo (2024). "Wittgenstein, probability and supraclassical logics". In: *[Manuscript submitted for publication[*.

Cuffaro, Michael E. (2010). "Wittgenstein on Prior Probabilities". In: *Proceedings of the Canadian Society for History and Philosophy of Mathematics* 23, pp. 85–98.

Figueiredo, Florian Franken (2023). "Wittgenstein's 1929–30 inquiries into probability". In: *Philosophical Investigations* 46.1, pp. 50–75.

Finetti, Bruno De (1931). *La logica dell'incerto*. Libreria Editrice Francesco Perrella.

— (1989). *La logica dell'incerto*. Il Saggiatore.

Foley, Richard (1992). "The Epistemology of Belief and the Epistemology of Degrees of Belief". In: *American Philosophical Quarterly* 29.2, pp. 111–124.

Hajek, Alan (2019). "Interpretations of Probability". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2019. Metaphysics Research Lab, Stanford University.

Halpern, Joseph Y. (2003). *Reasoning about Uncertainty*. Cambridge, MA, USA: MIT Press.

Hawthorne, James (2009). "The Lockean Thesis and the Logic of Belief". In: *Degrees of Belief*. Ed. by Franz Huber and Christoph Schmidt-Petri. Synthese Library: Springer, pp. 49–74.

Hay, Clare (2022). "Probability in Wittgenstein's Tractatus". In: *Philosophical Investigations* 45.2, pp. 130–145.

Kyburg, Henry Ely (1961). *Probability and the Logic of Rational Belief*. Middletown, CT, USA: Wesleyan University Press.

Leitgeb, Hannes (2017). *The Stability of Belief: How Rational Belief Coheres with Probability*. Oxford University Press.

Makinson, D. (2005). *Bridges from classical to nonmonotonic logic*. Lightning Source, Milton Keynes.

Ongaro, Malvina (2021). "The Interpretation of Probability in the Tractatus Logico-Philosophicus". In: *Wittgenstein-Studien* 12.1, pp. 131–144.

Paris, J. B. (1995). *The Uncertain Reasoner's Companion: A Mathematical Perspective*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press.

Piazza, M. and G. Pulcini (Oct. 2016). "Uniqueness of axiomatic extensions of cut-free classical propositional logic". In: *Logic Journal of the IGPL* 24.5, pp. 708–718.

Post, Emil (1990). “Introduction to a general theory of elementary propositions”. In: *From Frege to Gödel: A Source Book in Mathematical Logic*. Ed. by Jean vana Heijenoort. Harvard University Press.

Wittgenstein, Ludwig (1922). *Tractatus Logico-Philosophicus*. Routledge and Kegan Paul.

— (1971). *Prototractatus an Early Version of Tractatus Logico-Philosophicus. Edited by B.F. Mcguinness, T. Nyberg and G.H. Von Wright, with a Translation by D.F. Pears and B.F. Mcguinness. An Historical Introd. By G.H. Von Wright and a Facsim. Of the Author's Manuscript*. Cornell University Press.

— (2012). *The Big Typescript: TS 213, German English Scholars' Edition*. Blackwell Pub; Bilingual.

Wright, G. H. Von (1969). “Wittgenstein's views on probability”. In: *Revue Internationale de Philosophie* 23.88/89 (2/3), pp. 259–279.