Music and Logic: a connection between two worlds

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Abstract. *Music and mathematics have a long-standing relationship, but what about music and logic? Only recently have some authors started to explore the relationship between logic and music analysis, thanks to developments in both fields. The aim of this paper is to analyze this relationship, by developing a system capable of analyzing chord sequences using a logical presentation as well as create new harmonic structures. The logical presentation draws heavily from proof theory and its dual, i.e. tableuax. Also if music is not a proof, its adaptability makes it effective for this purpose. The attempt here proposed will try to apply proof theory to a brief, but important part of music: chord sequence analysis.*

Keywords: Music Analysis; Logic; Proof theory; Chords Analysis

1 Introduction

Logic is primarily used in mathematics to formalize human reasoning, and the study of the relationship between mathematics and music has a long-standing tradition. This work aims to explore the connection between logic and music, which has not been studied extensively. Specifically, the idea arose from my personal interest in proof theory and its ability to simplify complex sentences into simpler propositions. The objective of this paper is to make a preliminary attempt in this direction, by exploring the possibility of applying proof theory techniques to chord analysis.

The method presented in this paper in inspired by Neo-Riemannian and Tonfeld theories, which are systematic approaches to the musical structures' analysis, albeit not in a formal logical sense. The main goal of this paper is to introduce a rule-based logic method for chord analysis, which shares some similarities with structural proof theory (e.g., Troelstra's Basic Proof Theory [11]) in its logical foundations.

Something similar to this method has been presented in various articles, such as Rohrmeier: extended harmony [9], Granroth-Wilding, Mark and Steedman, Mark: Statistical Parsing for Harmonic Analysis of Jazz Chord Sequences [3] and Satoshi: modal logic music [10], but this contribution takes a different approach. Instead of decomposing a chord progression through grammar syntax, it presents a set of rules that can reduce the total number of chords, making the analysis simpler.

The method is as follows:

- we begin with a given chord progression;
- we apply rules that can *reduct* a certain set of chords into a smaller one;
- repeat the process until no further reductions are possible.

The corresponding method in proof theory is the decomposition of a proposition into a set of simpler ones to understand easier if the proposition is a tautology or not. But in the case of music analysis the final set of chords, that can't be simplified, will be called the *core set* and it will be associated with a particular type of chord set according to Tonfeld theory. The rules, once explained, can be analyzed both bottom-up and top-down, revealing the application of each rule at each level of the decomposition. The invertibility of each rule can also be used to compose new chord progressions, providing a mechanical method for choosing between different chords.

The paper is organized as follows: in the second section, we provide a description of Tonfeld and the Circle of Fifths; in the third section, we present our motivations for choosing structural proof theory and a very brief presentation of it is given; in the fourth section, we present the method for analysis in detail; in the fifth section, we outline the inverse process of composition using the rule-based method. Throughout the paper, we provide examples to help illustrate the motivations and the methods being discussed.

2 Tonfeld and circle of fiths

The Tonfeld theory [7] provides a visual depiction of the relationships between chords in tonal harmony using an infinite plane graph where each node represents a unique pitch class. Notes are understood as points in the graph, and chords are depicted by their relationships. The theory identifies three fundamental types of relationships: octatonic, hexatonic, and stacks of fifths, which are cyclic and sufficient to describe all other cyclic groups.

Instead of explicitly defining the octatonic and hexatonic sets for each note (see for example [8]), it is possible to specify only three octatonic and four hexatonic sets, because the others are permutations of these, thanks to the limited transposition modes ([6]). This allows for a more efficient and compact representation of the harmonic relations within the Tonfeld theory:

$$Oct_0 = \{C, D\flat, E\flat, E\natural, G\flat, G\natural, A, B\flat\} \tag{1}$$

$$Oct_1 = \{D\flat, D, E\natural, F, G, A\flat, B\flat, C\flat\} \tag{2}$$

$$Oct_2 = \{C, D, E\flat, F, G\flat, A\flat, A\natural, B\} \tag{3}$$

$$Hex_0 = \{C, E\flat, E, G, A\flat, B\} \tag{4}$$

$$Hex_1 = \{C\sharp, E, F, G\sharp, A, C\} \tag{5}$$

$$Hex_2 = \{D, F, F\sharp, A, B\flat, C\sharp\} \tag{6}$$

$$Hex_3 = \{E\flat, F\sharp, G, B\flat, B\natural, D\} \tag{7}$$

The stack of fifths is not a mode of limited transposition, so it is necessary to enumerate them for each note and for each expansion. This means that all possible stacks

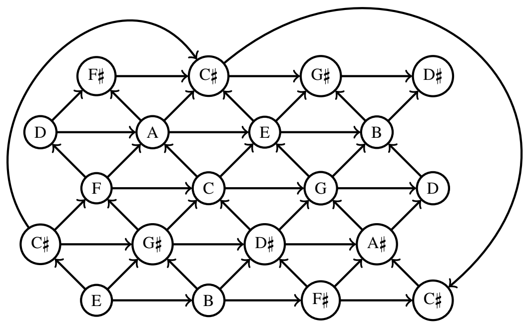


Fig. 1. A part of the Tonfeld. The lines outside the figure outline the relation between the same chords throughout the plane.

of fifths need to be explicitly listed, unlike the octatonic and hexatonic sets which can be represented with a little number of sets due to the limited transposition modes.

$$Fif_{C,1} = \{C, G\} \tag{8}$$

$$Fif_{C,2} = \{C, G, D\} \tag{9}$$

$$\vdots$$

This enumeration of every component, as is well-known, can be easily deduced thanks to the circle of fifths. Furthermore, this system simplifies the work with the proof-theoretic platform that will be presented later.

3 Why structural proof theory?

Proof theory is a branch of mathematical logic that studies the nature of mathematical proofs and their properties. The central questions in proof theory concern the nature of proof, the relationship between syntax and semantics, and the role of proofs in the development of mathematics. It is a fundamental component of mathematical logic, and has important applications in computer science, philosophy, and other fields. Proof theory aims to understand the nature of formal systems and develop techniques for analyzing and manipulating them; there are several approaches to proof theory, but the one we want to emphasize here is *structural proof theory*.

The origins of structural proof theory [11] can be traced back to the early 1930s when Gerhard Gentzen (1909-1945) introduced the concept in his doctoral thesis titled “Untersuchungen über das logische Schließen” [4]. in 1933. In this thesis, Gentzen presented two primary systems of logical rules: natural deduction and sequent calculus.

The former system aimed to closely align with the way theorems are typically proven in practice, while the latter system provided the framework through which Gentzen arrived at his main finding, often referred to as Gentzen's "Hauptsatz". This theorem states that any proof in classical logic can be transformed into a specific "cut-free" form, which means that the proof can be obtained without detours. Additionally, the cut-free proof has the subformula property, which states that all the premises used in the proof are contained in the conclusions. From this, general conclusions about proofs can be drawn, such as the consistency of the system of rules. The method has the following structure: the top formulas, also called *leaves*, represent the *starting point* of the proof ($q \vdash q$ and $p \vdash p$), while on the bottom we find the proven formula (i.e., $\vdash (p \wedge q) \to (q \wedge p)$).

$$\dfrac{\dfrac{\dfrac{\dfrac{q \vdash q \qquad p \vdash p}{q, p \vdash q \wedge p}\,(\wedge_{R,I})}{p, q \vdash q \wedge p}\,(ex.)}{p \wedge q \vdash q \wedge p}\,(\wedge_{L,I})}{\vdash (p \wedge q) \to (q \wedge p)}\,(\to_{R,I})$$

In this context, R and L denote the side of the rule to be applied; I represents the *introduction* of a rule; $ex.$ represents the exchange rule, which enables swapping of the terms in the proof. The symbol $\wedge$ represents conjunction, which can be interpreted in English as *and*, and $\to$ represents implication, interpreted as *if... then...*

This exposition, albeit brief, serves the purpose of highlighting the reasons why it is applied in both harmonic analysis and composition. The rules presented here, including the introduction of implications, are clear and allow for a visual representation of the steps involved in the proof. We believe that this system's clarity and duality make it an effective way to represent not only propositions but also chords. In proof theory, it is possible to use not only trees like the one presented earlier but also trees constructed *from the bottom* known as *tableaux*. These trees are constructed from the proposition to be proved, as shown below:

$$\begin{array}{c}
\vdash (p \wedge q) \to (q \wedge p) \\
| \\
p \wedge q \vdash q \wedge p \\
| \\
p,\ q \vdash q \wedge p \\
\diagup \qquad \diagdown \\
p,\ q \vdash q \qquad p,\ q \vdash p
\end{array}$$

The dual approach of the system, allowing for progression from the axioms or the propositions, will be useful in presenting the two-fold perspective we aim to convey in this article: analyzing chords from one direction and creating new harmonic structures from the other.

4 Rule based presentation

In proof theory, specifically in the style developed by Gentzen [11], a set of rules is used to introduce and eliminate certain logical connectives ($\wedge, \vee, \rightarrow, \neg$) in order to determine whether the topmost nodes of a proof correspond to axioms, i.e., whether $p \vdash p$ (*p proves p*). The purpose of this section is to provide a more systematic account of the application of certain harmonic rules, using the rules of harmony, the Tonfeld theory, and a proof-theoretic framework.

While music cannot be proven (it is not a proof), a set of fundamental rules can still be outlined and adapted by adding or removing rules. This article focuses on jazz tonal harmony and presents a construction using a limited set of rules, but it can be easily expanded.

Our attempt is to find a way to combine the generative theory of tonal music (GTTM) [9] with the ability of proof theory to explicitly indicate when and where a certain rule must or can be applied. This aims to provide a more systematic and logical approach to understanding and analyzing harmony in tonal music. It must be stressed that this attempt will not adhere to all of the structural rules typically found in proof theory. In fact, the only structural rule that we will use is the Contraction Rule, which will play a crucial role.

$$\frac{\vdash p, p}{\vdash p}\ (Contraction)$$

but there is no place for weakening, because we don't want that new chords can appear spontaneously:

$$\frac{\vdash p}{\vdash p, q}\ (Weakening)$$

and exchange, because we don't want that chords change position:

$$\frac{\vdash p, q}{\vdash q, p}\ (Exchange)$$

4.1 The first rules

In a Gentzen's style presentation a rule has the following form:

$$\frac{\vdash p \qquad \vdash q}{\vdash p \wedge q}\ (\wedge_I)$$

where the letter *I*, indicates the *introduction* rule. The reason for erasing a chord in the presented version of the rule is to identify the essential components of the harmonic sequence during harmonic analysis. Therefore, the objective is to isolate the *core* set of harmonies. In fact here the presentation is as follows:

Authentic Cadence:

$$\frac{\text{V7} \qquad \text{I}_{\text{MA}}{}^{7}}{\text{I}_{\text{MA}}{}^{7}}\ (\mathit{Fif}_{I,1})$$

This kind of cadence can be expanded with the stack of fifths:

$$\frac{\text{IIm7} \qquad \text{V7} \qquad \text{I}_{\text{MA}}{}^{7}}{\text{I}_{\text{MA}}{}^{7}}\ (\mathit{Fif}_{I,2})$$

$$\frac{\text{VIm7} \qquad \text{IIm7} \qquad \text{V7} \qquad \text{I}_{\text{MA}}{}^{7}}{\text{I}_{\text{MA}}{}^{7}}\ (\mathit{Fif}_{I,3})$$

$$\vdots$$

Another rule is the *Plagal Cadence*, which is extensively used in ancient as well as modern pop music. *Plagal Cadence* is a type of cadence that goes from the fourth scale degree to the first one. It can be schematized in three main ways, to also explicitly show the movement from the fourth minor scale degree to the first one. It can be interpreted as a particular case of the circle of fifths: from the last instances to the first one.

Plagal cadence :

$$\frac{\text{IV}_{\text{MA}}{}^{7} \qquad \text{I}_{\text{MA}}{}^{7}}{\text{I}_{\text{MA}}{}^{7}}\ \text{P.C.} \quad \frac{\text{IVm7} \qquad \text{I}_{\text{MA}}{}^{7}}{\text{I}_{\text{MA}}{}^{7}}\ \text{P.C.} \quad \frac{\text{IV}_{\text{MA}}{}^{7} \qquad \text{IVm7} \qquad \text{I}_{\text{MA}}{}^{7}}{\text{I}_{\text{MA}}{}^{7}}\ \text{P.C.}$$

To explicitly explain the other types of cadences, like the *Deceptive Cadence*, it must be noted that the tonic and the submediant have a lot of notes in common, which allows for their mutual substitution. For example, an authentic cadence can be transformed into a deceptive one by substituting the tonic with the submediant. Something similar occurs between the tonic and the mediant.

Inversions:

$$\frac{\text{VI}}{\text{I}}\ (i.) \qquad \frac{\text{III}}{\text{I}}\ (i.) \qquad \frac{\text{I}}{\text{III}}\ (i.) \qquad \frac{\text{I}}{\text{VI}}\ (i.)$$

In jazz and classical music, another rule is deduced: the tritone substitution. This rule allows for the substitution of a dominant chord with its relative tritone. This is particularly useful in the context of jazz improvisation and the creation of complex harmonic progressions in classical music. The tritone substitution adds more dissonance to the progression, and is one of the most important feature of jazz harmony. It can be used to create tension and dissonance and it is an essential tool to understand the harmonic language of jazz.

$$\frac{\text{V}}{\text{I}\sharp}\ (\mathit{tr})$$

From the Authentic Cadence and its inversions, it is possible to also obtain the Deceptive Cadence and the Authentic Cadence with the subdominant scale degree instead of the supertonic:

– Deceptive Cadence: V-VI;
– Authentic Cadence with subdominant: IV-I;

These variations help to create a more rich and complex harmonic language and can be used to create a different emotional or stylistic effect in the music.

$$\dfrac{\mathrm{V7} \qquad \dfrac{\mathrm{VIm7}}{\mathrm{I_{MA}}^7}\,(i)}{\mathrm{I_{MA}}^7}\,(Fif_{I,2})$$

$$\dfrac{\dfrac{(\mathrm{IV_{MA}}^7)}{\mathrm{IIm7}}\,(i.) \qquad \mathrm{V7} \qquad \mathrm{I_{MA}}^7}{\mathrm{I_{MA}}^7}\,(Fif_{I,2})$$

4.2 Examples

Example 1. The following example is taken from "But not for me" by George Gershwin. The structure is divided into sections, because the tree was too long.

$$\dfrac{\dfrac{\mathrm{F7} \quad \mathrm{B\flat 7} \quad \mathrm{E\flat MA}^7}{\mathrm{E\flat MA}^7}\,Fif_{E\flat,3} \qquad \dfrac{\mathrm{Gm7}}{\mathrm{E\flat MA}^7}\,(i.) \qquad \dfrac{\mathrm{F7} \quad \mathrm{B\flat 7} \quad \mathrm{E\flat MA}^7}{\mathrm{E\flat MA}^7}\,Fif_{E\flat,3}}{\mathrm{E\flat MA}^7}$$

$$\dfrac{\dfrac{\dfrac{\mathrm{B\flat m7} \quad \mathrm{E\flat 7} \quad \mathrm{A\flat MA}^7}{\mathrm{A\flat MA}^7}\,Fif_{A\flat,3} \qquad \dfrac{\dfrac{\mathrm{D\flat 7}}{\mathrm{Gm7}}\,(tr.)}{\mathrm{E\flat MA}^7}\,(i.) \qquad \mathrm{E\flat MA}^7}{\mathrm{E\,\flat MA}^7}\,P.C._{E\flat,3} \qquad \dfrac{\mathrm{Fm7} \quad \mathrm{B\flat 7}}{\mathrm{B\flat 7}}\,Fif_{B\flat,2}}{\mathrm{E\flat MA}^7 \quad \mathrm{B\flat 7}}$$

$$\dfrac{\dfrac{\dfrac{\mathrm{B\flat m7} \quad \mathrm{E\flat 7} \quad \mathrm{A\flat MA}^7}{\mathrm{A\flat MA}^7}\,Fif_{A\flat,3} \qquad \dfrac{\dfrac{\mathrm{D\flat 7}}{\mathrm{Gm7}}\,(tr.)}{\mathrm{E\flat MA}^7}\,(i.) \qquad \mathrm{E\flat MA}^7}{\mathrm{E\,\flat MA}^7}\,P.C._{E\flat,3} \qquad \dfrac{\mathrm{Fm7} \quad \mathrm{B\flat 7} \quad \mathrm{E\flat MA}^7}{\mathrm{E\flat MA}^7}\,Fif_{E\flat,3}}{\mathrm{E\flat MA}^7}$$

As we can see the main chord is $\mathrm{E\flat MA}^7$, that is always the main chord.

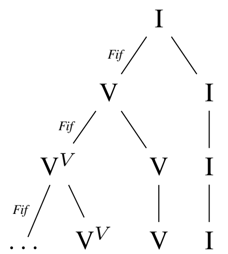
5 Composition of new musical structures through a rule based presentation

This method uses invertible rules to create novel harmonic tonal structures. The process is straightforward and involves selecting a fundamental node, determining the desired length of the structure, and applying the rules to expand the system until the required number of chords is reached. By using this systematic and logical approach based on the invertibility of the rules, it's possible to compose original harmonies.

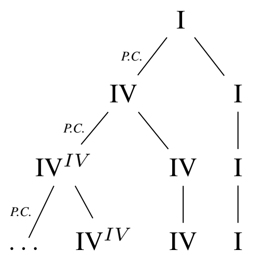
5.1 Tableux

The inversion of the rules made in section 4.1 can be inverted thanks to the dual of proof theory: tableux.

Generation of the Authentic Cadence:

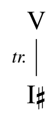


Generation of the Plagal Cadence:

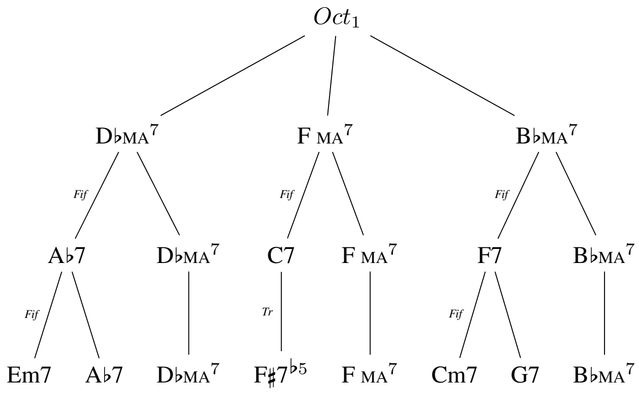


The inversions:

Triton substitution:



Example 2. Firstly, we'll form a basic arrangement of three notes taken from the initial Octatonic set. Then, we'll establish a preferred duration for the arrangement, say 8 measures. We'll utilize various techniques, such as stack of fifths, plagal cadence, and tritone substitution, to extend the pattern following certain guidelines until we attain the desired number of chords. Consequently, we'll obtain 11 distinctive chords, resulting in an 8-measure arrangement with increased intricacy and musical appeal.



This way of composition can be automatized to create new and different harmonic structures, always remaining into a tonal configuration, but what if we want to create a non-tonal structure?

5.2 New rules

One of the interesting thing about our system is that it is possible to add new rules. Music and harmony, in fact, can change between ages and the rule-based system can be expanded with new rules if they are considered useful for a certain kind of analysis or a certain kind of composition. Suppose that the analyzed piece is taken from the baroque period and so it is important to explicit the *picardy third*. Then it is easy to add a rule that could be something like:

$$\frac{\text{Im} \qquad \text{V} \qquad \text{I}}{\text{Im} \qquad \text{I}}\ \textit{P.T.}$$

This could seem redundant in respect of the rule of the stack of fifths, but the attempt here is to create something general that could be useful also in specific cases. In the baroque chorals, for example, understand when there is an authentic minor cadence or a picardy third could be useful, because a picardy third indicates the end of a phrase or of the piece.

Example 3. Let's consider, for example, J.S. Bach's Jesu, meine Freude (figure 2), this is an example of picardy third.

To analyze these bars it is possible to use the new rule:

$$\frac{\dfrac{\text{Em} \qquad \text{Am} \qquad \text{Em}}{\text{Em}}\ \textit{P.C.} \qquad \dfrac{\text{F}\sharp 7 \qquad \text{B7} \qquad \text{E}}{\text{E}}\ \textit{P.T.}}{\text{Em} \qquad \text{E}}$$

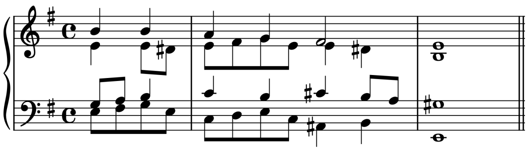
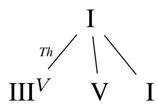


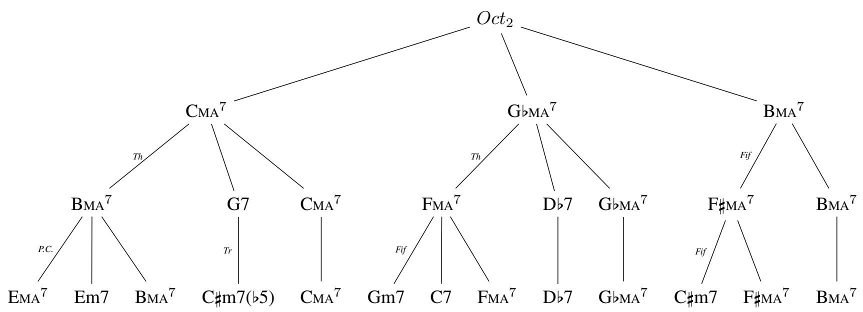
Fig. 2. J.S. Bach's Jesu, meine Freude; mm. 11-13

New rules in composition In addition to their traditional use in tonality, add rules can also be employed in composition to generate unconventional results. For instance, a new rule could be introduced to mandate the inclusion of the third degree of a chord in any dominant chord that appears. This rule might read as follows: "Whenever a dominant chord is encountered, it must contain also the third degree" and let's call that *Th.*.



Incorporating this additional layer would introduce an added level of intricacy and diversity to the harmonic arrangements produced by the system, leading to a greater potential for novel and unforeseen outcomes. It is crucial to acknowledge that the regulations you integrate will shape the final composition to align with your specific requirements.

Example 4. For example try to write a new harmonic form using this rule with also some other rule:



5.3 Proof theory method and CCG

The presented method shares similarities with the one presented in [3] that uses Combinatorial Category Grammar (CCG), but there are some notable differences. On one hand, the reduction method we propose is more flexible than the one presented in [3]. On the other hand, our method is not currently linked to machine learning or automatic analysis, which are areas that we plan to explore in the future. It's worth noting that the two methods are not in conflict and can ideally be combined in the future.

The contraction method we propose is particularly useful because it's malleable: we can add new rules to analyze and stress different musical aspects, and we can even invert the method to create new harmonic structures. In contrast, the method presented in [3] relies on a statistical machine learning technique that may not be as straightforward to implement using our proof theoretic presentation. Moreover, our proof theoretic presentation can be inverted to create new harmonic structures, which CCG can only achieve by working on the rules. However, this inversion process is not as straightforward as it is with our method.

A common point between CCG and our method is the philosophical and musicological idea that chord progressions are driven by the listener's *expectations* of progression, based on the same harmonic Riemannian concept. However, our method can be expanded due to its ability to incorporate new rules.

6 Conclusions

The presented method is a different approach to understanding chord progressions, drawing inspiration from proof theory. While music cannot be proven, this rule-based method simplifies and clearly demonstrates the invertibility of the rules between analysis and composition. A method has been introduced to simplify harmonic structures into fundamental chords, and an inverse method can create new harmonic structures based on established rules. The approach's advantage is that new rules can be added to the system to emphasize specific structures or introduce new ones.

This approach to harmonic analysis, based on structural proof theory, offers several advantages for students and professionals alike. Firstly, its visual representation can aid in better understanding the underlying principles of harmony. By breaking down complex harmonic structures into their component parts and representing them as a tree-like structure, students can more easily grasp the relationships between chords and the rules that govern them. Secondly, this approach can be helpful in promoting creativity when creating new harmonic structures by providing the opportunity to choose new rules and leading to unexpected solutions and unique harmonic progressions. Lastly, this method can help shed light on a particular harmonic structure that may otherwise go unnoticed. By approaching a harmonic structure several times, each time with a different lens, an analyst can highlight a particular composer's choice over another, revealing the nuances and subtleties of their harmonic language. Overall, this approach to harmonic analysis offers a powerful tool for understanding and creating harmonic structures, and its potential applications are wide-ranging, from the classroom to the automatic composition.

Further research could expand the rule set and apply different rules to various musical genres and historical periods. The method could be formalized for analyzing music structures using tools like Open Music. Additionally, this application could be inverted to create an automatic composer of harmonic structures. In a future it will, hopefully, also interesting to use this method to analyse harmonic structures thanks to Machine Learning.

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